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## CHARACTERIZATIONS OF BERGMAN SPACES AND BLOCH SPACE IN THE UNIT BALL OF $\mathbb{C}^n$

CAIHENG OUYANG, WEISHENG YANG, AND RUHAN ZHAO

**ABSTRACT.** In this paper we prove that, in the unit ball  $B$  of  $\mathbb{C}^n$ , a holomorphic function  $f$  is in the Bergman space  $L_a^p(B)$ ,  $0 < p < \infty$ , if and only if

$$\int_B |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} (1 - |z|^2)^{n+1} d\lambda(z) < \infty,$$

where  $\tilde{\nabla}$  and  $\lambda$  denote the invariant gradient and invariant measure on  $B$ , respectively. Further, we give some characterizations of Bloch functions in the unit ball  $B$ , including an exponential decay characterization of Bloch functions. We also give the analogous results for  $BMOA(\partial B)$  functions in the unit ball.

### 1. INTRODUCTION

Let  $A(B)$  denote the class of holomorphic functions in the unit ball of  $\mathbb{C}^n$ . For  $0 < p < \infty$ , the Bergman spaces  $L_a^p(B)$ , the Hardy spaces  $H^p(B)$  and the Bloch space  $\mathcal{B}(B)$  on the unit ball  $B$  are defined respectively as

$$L_a^p(B) = \left\{ f : f \in A(B), \|f\|_{L_a^p}^p = \int_B |f(z)|^p dm(z) < \infty \right\},$$

$$H^p(B) = \left\{ f : f \in A(B), \|f\|_{H^p}^p = \sup_{0 < r < 1} \int_{\partial B} |f(r\xi)|^p d\sigma(\xi) < \infty \right\}$$

and

$$\mathcal{B}(B) = \left\{ f : f \in A(B), \|f\|_{\mathcal{B}} = \sup_{z \in B} Q_f(z) < \infty \right\},$$

where  $Q_f$  was defined by R. Timoney in [9],  $dm$  is the normalized Lebesgue measure on  $B$ , and  $d\sigma$  is the normalized Lebesgue measure on the boundary  $\partial B$  of  $B$ .

In [8], M. Stoll proved that a holomorphic function  $f$  on  $B$  is in  $H^p(B)$ ,  $0 < p < \infty$ , if and only if

$$\int_B |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} (1 - |z|^2)^n d\lambda(z) < \infty,$$

where  $\tilde{\nabla}$  denotes the invariant gradient and  $\lambda$  the invariant measure on  $B$ . Furthermore, if  $f \in H^p(B)$ , then

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$$\lim_{r \rightarrow 1} (1 - r^2)^n \int_{B_r} |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} d\lambda(z) = 0,$$

where  $B_r = \{z \in B : |z| < r\}$ .

These results were first given by S. Yamashita in [11] and [12] in the unit disk of  $\mathbb{C}$ . In [13], the results for Bergman spaces similar to that of Yamashita's were given on the unit disk  $D$ .

The main purpose of this paper is to obtain the analogous result for functions in the Bergman spaces  $L_a^p$  on the unit ball  $B$  of  $\mathbb{C}^n$ . Furthermore, some new characterizations of Bloch space  $\mathcal{B}(B)$ , including an exponential decay type characterization, are given too. The main results of this paper, which are also similar to that of [13] in case  $n = 1$ , are as follows:

**Theorem 1.** *A holomorphic function  $f$  is in  $L_a^p(B)$ ,  $0 < p < \infty$ , if and only if*

$$\int_B |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} (1 - |z|^2)^{n+1} d\lambda(z) < \infty.$$

Furthermore, if  $f \in L_a^p(B)$ , then

$$\lim_{r \rightarrow 1} (1 - r^2)^{n+1} \int_{B_r} |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} d\lambda(z) = 0.$$

**Theorem 2.** *Let  $n > 1$ ,  $p \geq 2$ ; then the following quantities are equivalent:*

- (i)  $\|f\|_{\mathcal{B}}^p$ ,
- (ii)  $J_2 = \sup_{a \in B} \int_B |\tilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} \cdot (1 - |\varphi_a(z)|^2)^{n+1} |\varphi_a(z)|^{-2n+2} d\lambda(z)$ ,
- (iii)  $J_3 = \sup_{a \in B} \int_B |\tilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} [G(z, a)]^{1+\frac{1}{n}} d\lambda(z)$ ,

where  $\varphi_a$  denotes the involutive automorphism of  $B$  satisfying  $\varphi_a(0) = a$ ,  $\varphi_a(a) = 0$ ,  $\varphi_a(\varphi_a(z)) = z$ , and  $G(z, a)$  denotes the Green's function of  $B$ .

**Theorem 3.** *Let  $n > 1$ ; then a holomorphic function  $f \in \mathcal{B}(B)$  if and only if for every  $a \in B$  and every  $t > 0$  there are positive constants  $K$  and  $\beta$ , such that*

$$\int_{E_{a,t}} |\tilde{\nabla} f(z)|^2 [G(z, a)]^{1+\frac{1}{n}} d\lambda(z) \leq Ke^{-\beta t},$$

where  $E_{a,t} = \{z \in B : |f(z) - f(a)| > t\}$ . When  $f \in \mathcal{B}$ ,  $K = K_0 \|f\|_{\mathcal{B}}^2$ ,  $\beta = C / \|f\|_{\mathcal{B}}$ , where  $K_0$  and  $C$  are constants depending only on  $n$ .

In Section 2, we first give some notations. Theorem 1 is proved in Section 3. Theorems 2 and 3 are proved in Section 4. In Section 5, we give some characterizations of  $BMOA(\partial B)$  which are similar to Theorems 2 and 3.

## 2. NOTATIONS

For each  $a \in B$ , let  $\varphi_a(z)$  denote the involutive automorphism of  $B$  as given in [6] by W. Rudin. Let  $\nabla f(z) = (\partial f / \partial z_1, \dots, \partial f / \partial z_n)$  denote the complex gradient of  $f$  and  $Rf = \sum_{j=1}^n z_j (\partial f / \partial z_j)$  the radial derivative of  $f$ . Let

$$d\lambda(z) = \frac{n + 1}{(1 - |z|^2)^{n+1}} dm(z);$$

then  $d\lambda$  is the invariant volume measure corresponding to the Bergman metric on  $B$ ; that is,

$$\int_B f(z) d\lambda(z) = \int_B f \circ \psi(z) d\lambda(z)$$

for each  $f \in L^1(d\lambda)$  and all  $\psi \in \mathcal{M}$ , the group of Möbius transformations of  $B$ .

For  $f \in C^2(\Omega)$ ,  $\Omega$  an open subset of  $B$ , define

$$\tilde{\Delta}f(z) = \frac{1}{n+1} \Delta(f \circ \varphi_z)(0),$$

as in [1],

$$\tilde{\Delta}f(z) = \frac{4}{n+1} (1 - |z|^2) \sum_{i,j=1}^n (\delta_{ij} - z_i \bar{z}_j) \frac{\partial^2 f(z)}{\partial z_i \partial \bar{z}_j}.$$

The operator  $\tilde{\Delta}$  is invariant under  $\mathcal{M}$ ; that is,  $\tilde{\Delta}(f \circ \psi) = (\tilde{\Delta}f) \circ \psi$  for all  $\psi \in \mathcal{M}$ . See [6, Section 4.1] for details. Let  $\tilde{\nabla}$  denote the invariant gradient on  $B$ . Then

$$(\tilde{\nabla}f(z), \tilde{\nabla}g(z)) = \frac{4}{n+1} (1 - |z|^2) \mathcal{R} \left[ \sum_{i,j=1}^n (\delta_{ij} - z_i \bar{z}_j) \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial \bar{z}_j} \right].$$

If  $f$  is holomorphic on  $B$ , it is given in [8] that

$$\tilde{\Delta}|f|^2 = |\tilde{\nabla}f|^2 = \frac{4}{n+1} (1 - |z|^2) (|\nabla f|^2 - |Rf|^2).$$

Throughout this paper,  $C$  and  $C_j$  are constants depending only on the dimension  $n$ .  $M$  is a finite number, and  $M(r)$  is a finite number for a fixed  $r \in (0, 1]$ .  $C$  is not necessarily the same in each appearance, nor are  $C_j$ ,  $M$ ,  $M(r)$ .

For convenience,  $A(f, r) \sim B(f, r)$  means that there exist constants  $N_1, N_2, C_1$  and  $C_2$ , so that

$$N_1 + C_1 A(f, r) \leq B(f, r) \leq N_2 + C_2 A(f, r),$$

where  $N_1, N_2$  may depend on  $f$ , but they are finite quantities for a fixed function  $f$ .

By [10], the invariant Green's function on  $B$  is given by  $G(z, a) = g(\varphi_a(z))$ , where

$$g(z) = \frac{n+1}{2n} \int_{|z|}^1 (1-t^2)^{n-1} t^{-2n+1} dt.$$

Here we state the Green's formula for an invariant Laplacian (see [7, (92.5)]). If  $\Omega$  is an open subset of  $B$ ,  $\bar{\Omega} \subset B$ , whose boundary is good enough (in our application,  $\Omega$  will be an annulus) and if  $u, v$  are real-valued functions such that  $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ , then

$$\int_{\Omega} (u \tilde{\Delta}v - v \tilde{\Delta}u) d\bar{\tau} = \int_{\partial\Omega} \left( u \frac{\partial v}{\partial \bar{n}} - v \frac{\partial u}{\partial \bar{n}} \right) d\bar{\sigma},$$

where  $\bar{\tau}$  and  $\bar{\sigma}$  are the volume element on  $B$  and surface area element on  $\partial\Omega$  determined by the Bergman metric, and  $\frac{\partial}{\partial \bar{n}}$  denotes the outward normal

differentiation along  $\partial\Omega$  with respect to the Bergman metric. It is known (cf. [1]) that the volume element  $\bar{\tau}$  is given by

$$d\bar{\tau}(z) = \frac{\omega_n(n+1)^n}{2n(1-|z|^2)^{n+1}} dm(z),$$

where  $\omega_n$  denotes the Euclidean surface area of  $\partial B$  and the surface area element  $\bar{\sigma}_r$  on  $\partial B_r$  is given by

$$d\bar{\sigma}_r(r\xi) = \frac{\omega_n(n+1)^{n-\frac{1}{2}}r^{2n-1}}{(1-r^2)^n} d\sigma(\xi).$$

### 3. PROOF OF THEOREM 1

To prove Theorem 1, for  $\varepsilon > 0$ , let

$$v_\varepsilon(z) = (|f(z)|^2 + \varepsilon)^{p/2}, \quad 0 < p < \infty;$$

then  $v_\varepsilon \in C^\infty$ . Since  $\tilde{\Delta}g = 0$  on  $B \setminus \{0\}$  and  $g = g(r)$  on  $\partial B_r = \{z \in B : |z| = r\}$ , using Green's formula with  $u = g - g(r)$ ,  $v = v_\varepsilon$  and  $\Omega = \{z \in B : \delta < |z| < r\}$ , we can conclude

$$\begin{aligned} & \int_{\delta < |z| < r} (g - g(r))\tilde{\Delta}v_\varepsilon d\bar{\tau} \\ &= - \int_{\partial B_r} v_\varepsilon \frac{\partial g}{\partial \bar{n}} d\bar{\sigma}_r - \left[ (g(\delta) - g(r)) \int_{\partial B_\delta} \frac{\partial v_\varepsilon}{\partial \bar{n}} d\bar{\sigma} - \int_{\partial B_\delta} v_\varepsilon \frac{\partial g}{\partial \bar{n}} d\bar{\sigma}_\delta \right]. \end{aligned}$$

Because  $\frac{\partial v_\varepsilon}{\partial \bar{n}}$  is bounded on  $\partial B_\delta$ ,  $g(\delta)\delta^{2n-1} \rightarrow 0$  ( $\delta \rightarrow 0$ ) and  $g$  is integrable near 0 ( $\delta \neq 0$ ), taking the limit  $\delta \rightarrow 0$ , we get

$$(1) \quad \int_{B_r} (g - g(r))\tilde{\Delta}v_\varepsilon d\lambda = \int_{\partial B} v_\varepsilon(r\xi) d\sigma(\xi) - v_\varepsilon(0).$$

Let

$$f_p^\#(z) = \frac{p^2}{4} |f(z)|^{p-2} |\tilde{\nabla} f(z)|^2,$$

by [8], for  $0 < p < \infty$ ; when  $\varepsilon \rightarrow 0$ ,

$$\tilde{\Delta}v_\varepsilon(z) \rightarrow f_p^\#(z) \quad \text{a.e. on } B.$$

For a fixed  $r$ , from (1) and by the monotone convergence theorem, we have

$$(2) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{B_r} (g - g(r))\tilde{\Delta}v_\varepsilon d\lambda &= \lim_{\varepsilon \rightarrow 0} \left( \int_{\partial B} v_\varepsilon(r\xi) d\sigma(\xi) - v_\varepsilon(0) \right) \\ &= \int_{\partial B} |f(r\xi)|^p d\sigma(\xi) - |f(0)|^p. \end{aligned}$$

By (2) and the Fatou Lemma

$$\begin{aligned} \int_{B_r} (g - g(r))f_p^\# d\lambda &= \int_{B_r} \liminf_{\varepsilon \rightarrow 0} (g - g(r))\tilde{\Delta}v_\varepsilon d\lambda \\ &\leq \liminf_{\varepsilon \rightarrow 0} \int_{B_r} (g - g(r))\tilde{\Delta}v_\varepsilon d\lambda \\ &= \int_{\partial B} |f(r\xi)|^p d\sigma(\xi) - |f(0)|^p = M(r) < \infty; \end{aligned}$$

thus,  $(g - g(r))f_p^\#$  is integrable on  $B_r$  with respect to  $d\lambda$ .

As  $0 < p < 2$ , for a fixed  $r$ , by [8],  $\tilde{\Delta}v_\varepsilon(z) \leq \frac{2}{p}f_p^\#(z)$ , a.e. on  $B$ , and thus

$$(g - g(r))\tilde{\Delta}v_\varepsilon \leq (g - g(r))\frac{2}{p}f_p^\#, \quad \text{a.e. on } B.$$

As  $p \geq 2$ , for  $\varepsilon \in (0, 1]$

$$\tilde{\Delta}v_\varepsilon \leq M(r) < \infty, \quad \text{on } B_r,$$

and thus

$$(g - g(r))\tilde{\Delta}v_\varepsilon \leq M(r)(g - g(r))$$

(here  $g - g(r)$  is obviously integrable on  $B_r$ ).

Using the dominated convergence theorem with both  $0 < p < 2$  and  $p \geq 2$  and from (2), we get

$$\begin{aligned} (3) \quad \int_{B_r} (g - g(r))f_p^\# d\lambda &= \lim_{\varepsilon \rightarrow 0} \int_{B_r} (g - g(r))\tilde{\Delta}v_\varepsilon d\lambda \\ &= \int_{\partial B} |f(r\xi)|^p d\sigma(\xi) - |f(0)|^p. \end{aligned}$$

Let

$$\chi_{|z|}(t) = \begin{cases} 1, & t > |z|, \\ 0, & \text{otherwise;} \end{cases}$$

then the left side of (3) is

$$\begin{aligned} (4) \quad &\int_{B_r} (g(z) - g(r))f_p^\#(z) d\lambda(z) \\ &= \int_{B_r} f_p^\#(z) d\lambda(z) \left( \frac{n+1}{2n} \int_{|z|}^r \frac{(1-t^2)^{n-1}}{t^{2n-1}} dt \right) \\ &= \frac{n+1}{2n} \int_{B_r} f_p^\#(z) d\lambda(z) \int_0^r \frac{(1-t^2)^{n-1}}{t^{2n-1}} \chi_{|z|}(t) dt \\ &= \frac{n+1}{2n} \int_0^r \frac{(1-t^2)^{n-1}}{t^{2n-1}} dt \int_{B_t} f_p^\#(z) d\lambda(z). \end{aligned}$$

Obviously, the end of (4)

$$\begin{aligned} (5) \quad &\frac{n+1}{2n} \int_0^r \frac{(1-t^2)^{n-1}}{t^{2n-1}} dt \int_{B_t} f_p^\#(z) d\lambda(z) \\ &\geq \frac{n+1}{2n} r^{-2n+1} \int_0^r (1-t^2)^{n-1} dt \int_{B_t} f_p^\#(z) d\lambda(z). \end{aligned}$$

On the other hand, for  $0 < r < 1$ , there exists a positive integer  $k$ , so that  $1/2^k < r \leq 1/2^{k-1}$ ; then

$$\begin{aligned}
 & \frac{n+1}{2n} \int_0^r \frac{(1-t^2)^{n-1}}{t^{2n-1}} \left( \int_{B_t} f_p^\#(z) d\lambda(z) \right) dt \\
 &= \frac{n+1}{2n} \left( \int_0^{\frac{1}{2k}} + \int_{\frac{1}{2k}}^r \right) \frac{(1-t^2)^{n-1}}{t^{2n-1}} \left( \int_{B_t} f_p^\#(z) d\lambda(z) \right) dt \\
 &\leq \frac{n+1}{2n} \int_0^{\frac{1}{2}} \frac{(1-t^2)^{n-1}}{t^{2n-1}} \left( \int_{B_t} f_p^\#(z) d\lambda(z) \right) dt \\
 (6) \quad &+ \frac{n+1}{2n} 2^{k(2n-1)} \left( \frac{2^{-(k-1)}}{r} \right)^{2n-1} \int_0^r (1-t^2)^{n-1} \left( \int_{B_t} f_p^\#(z) d\lambda(z) \right) dt \\
 &= \frac{n+1}{2n} \int_0^{\frac{1}{2}} \frac{(1-t^2)^{n-1}}{t^{2n-1}} \left( \int_{B_t} f_p^\#(z) d\lambda(z) \right) dt \\
 &+ \frac{n+1}{2n} 2^{2n-1} r^{-2n+1} \int_0^r (1-t^2)^{n-1} \left( \int_{B_t} f_p^\#(z) d\lambda(z) \right) dt \\
 &= I_1 + I_2.
 \end{aligned}$$

By (4) and (3),

$$\begin{aligned}
 I_1 &= \int_{B_{\frac{1}{2}}} (g(z) - g(\tfrac{1}{2})) f_p^\#(z) d\lambda(z) \\
 &= \int_{\partial B} |f(\tfrac{1}{2}\xi)|^p d\sigma(\xi) - |f(0)|^p < \infty.
 \end{aligned}$$

Thus, it follows from (4), (5) and (6) that

$$\int_{B_r} (g(z) - g(r)) f_p^\#(z) d\lambda(z) \sim r^{-2n+1} \int_0^r (1-t^2)^{n-1} \left( \int_{B_t} f_p^\#(z) d\lambda(z) \right) dt.$$

Moreover, by (3), we have

$$(7) \quad \int_{\partial B} |f(r\xi)|^p d\sigma(\xi) \sim r^{-2n+1} \int_0^r (1-t^2)^{n-1} \left( \int_{B_t} f_p^\#(z) d\lambda(z) \right) dt.$$

Hence

$$\begin{aligned}
 \|f\|_{L^p_a}^p &= 2n \int_0^1 r^{2n-1} dr \int_{\partial B} |f(r\xi)|^p d\sigma(\xi) \\
 &\sim \int_0^1 dr \left( \int_0^r (1-t^2)^{n-1} dt \int_{B_t} f_p^\#(z) d\lambda(z) \right) \\
 &= \int_0^1 (1-t^2)^{n-1} dt \int_{B_t} f_p^\#(z) d\lambda(z) \int_t^1 dr \\
 &\sim \int_0^1 (1-t^2)^n dt \int_{B_t} f_p^\#(z) d\lambda(z) \\
 &= \int_B f_p^\#(z) d\lambda(z) \int_{|z|}^1 (1-t^2)^n dt \\
 &\sim \int_B (1-|z|^2)^{n+1} f_p^\#(z) d\lambda(z) \\
 &\sim \int_B |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} (1-|z|^2)^{n+1} d\lambda(z),
 \end{aligned}$$

and thus

$$f \in L_a^p(B) \Leftrightarrow \int_B |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} (1 - |z|^2)^{n+1} d\lambda(z) < \infty.$$

That is the main part of Theorem 1.

When  $f \in L_a^p(B)$ , by the above result,

$$\begin{aligned} & \int_0^1 \int_{B_t} |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} (1 - |z|^2)^n d\lambda(z) dt \\ &= \int_B \int_{|z|}^1 |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} (1 - |z|^2)^n dt d\lambda(z) \\ &\leq \int_B |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} (1 - |z|^2)^{n+1} d\lambda(z) < \infty, \end{aligned}$$

and thus

$$(8) \quad \lim_{r \rightarrow 1} \int_r^1 \int_{B_t} |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} (1 - |z|^2)^n d\lambda(z) dt = 0.$$

Furthermore

$$\begin{aligned} & (1 - r^2)^{n+1} \int_{B_r} |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} d\lambda(z) \\ (9) \quad & \leq 2(1 - r) \int_{B_r} |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} (1 - |z|^2)^n d\lambda(z) \\ & \leq 2 \int_r^1 \int_{B_t} |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} (1 - |z|^2)^n d\lambda(z) dt. \end{aligned}$$

By (8) and (9), we conclude

$$\lim_{r \rightarrow 1} (1 - r^2)^{n+1} \int_{B_r} |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} d\lambda(z) = 0.$$

That is the last part of Theorem 1.

*Remark 1.* From (7) we can get another proof of Theorem 1 of [8]. In fact, letting  $r \rightarrow 1$  in (7), we get

$$\begin{aligned} \|f\|_{H^p}^p &\sim \int_0^1 (1 - t^2)^{n-1} dt \int_{B_t} f_p^\#(z) d\lambda(z) \\ &= \int_B f_p^\#(z) d\lambda(z) \int_{|z|}^1 (1 - t^2)^{n-1} dt \\ &\sim \int_B |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} (1 - |z|^2)^n d\lambda(z). \end{aligned}$$

*Remark 2.* Theorem 2 in [8] can be concluded from (3). Taking the limit  $r \rightarrow 1$  on two sides of (3), using the monotone convergence theorem, we get

$$(10) \quad \|f\|_{H^p}^p = |f(0)|^p + \frac{p^2}{4} \int_B |\tilde{\nabla} f(z)|^2 |f(z)|^{p-2} g(z) d\lambda(z).$$

This is equivalent to the result of Theorem 2 of [8].



4. CHARACTERIZATIONS OF BLOCH SPACE IN THE UNIT BALL

**Lemma 1.** *Let  $n \geq 2$  be an integer, then there are constants  $C_1$  and  $C_2$  such that, for all  $z \in B \setminus \{0\}$ ,*

$$C_1(1 - |z|^2)^n |z|^{-2(n-1)} \leq g(z) \leq C_2(1 - |z|^2)^n |z|^{-2(n-1)},$$

where

$$g(z) = \frac{n + 1}{2n} \int_{|z|}^1 r^{-2n+1} (1 - r^2)^{n-1} dr.$$

*Proof.* It is easy to see that

$$(11) \quad \lim_{|z| \rightarrow 1} \frac{g(z)}{(1 - |z|^2)^n |z|^{-2(n-1)}} = \frac{n + 1}{4n^2},$$

and

$$(12) \quad \lim_{|z| \rightarrow 0} \frac{g(z)}{(1 - |z|^2)^n |z|^{-2(n-1)}} = \frac{n + 1}{4n(n - 1)}.$$

The result of Lemma 1 comes by the continuity of  $g(z)$ , (11) and (12).

*Proof of Theorem 2.* Replacing  $f$  in (3) by  $f_a = f \circ \varphi_a(\cdot) - f \circ \varphi_a(0)$ , we get

$$\begin{aligned} & \frac{p^2}{4} \int_{B_r} |\tilde{\nabla} f_a(w)|^2 |f_a(w)|^{p-2} (g(w) - g(r)) d\lambda(w) \\ &= \int_{\partial B} |f_a(r\zeta)|^p d\sigma(\zeta). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{4}{p^2} \|f_a\|_{L_a^p}^p &= \frac{4}{p^2} \int_B |f_a(z)|^p dm(z) = \frac{8n}{p^2} \int_0^1 r^{2n-1} dr \int_{\partial B} |f_a(r\zeta)|^p d\sigma(\zeta) \\ &= 2n \int_0^1 r^{2n-1} dr \int_{B_r} |\tilde{\nabla} f_a(w)|^2 |f_a(w)|^{p-2} (g(w) - g(r)) d\lambda(w) \\ &\leq 2n \int_0^1 dr \int_{B_r} |\tilde{\nabla} f_a(w)|^2 |f_a(w)|^{p-2} (g(w) - g(r)) d\lambda(w) \\ &= (n + 1) \int_0^1 dr \int_0^r \frac{(1 - t^2)^{n-1}}{t^{2n-1}} dt \int_{B_t} |\tilde{\nabla} f_a(w)|^2 |f_a(w)|^{p-2} d\lambda(w) \\ &= (n + 1) \int_0^1 \frac{(1 - t^2)^{n-1}}{t^{2n-1}} dt \int_t^1 dr \int_{B_t} |\tilde{\nabla} f_a(w)|^2 |f_a(w)|^{p-2} d\lambda(w) \\ &\leq (n + 1) \int_0^1 \frac{(1 - t^2)^n}{t^{2n-1}} dt \int_{B_t} |\tilde{\nabla} f_a(w)|^2 |f_a(w)|^{p-2} d\lambda(w) \\ &= (n + 1) \int_B |\tilde{\nabla} f_a(w)|^2 |f_a(w)|^{p-2} d\lambda(w) \int_{|w|}^1 \frac{(1 - t^2)^n}{t^{2n-1}} dt \\ &\leq 2n \int_B |\tilde{\nabla} f_a(w)|^2 |f_a(w)|^{p-2} (1 - |w|^2) g(w) d\lambda(w) \\ &\leq C \int_B |\tilde{\nabla} f_a(w)|^2 |f_a(w)|^{p-2} (1 - |w|^2)^{n+1} |w|^{-2n+2} d\lambda(w). \end{aligned}$$

The last inequality is given by Lemma 1. Letting  $\varphi_a(w) = z$ , we can find

$$\|f_a\|_{L_a^p}^p \leq C \int_B |\tilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} \cdot (1 - |\varphi_a(z)|^2)^{n+1} |\varphi_a(z)|^{-2n+2} d\lambda(z).$$

Thus we have

$$\sup_{a \in B} \|f_a\|_{L_a^p}^p \leq C J_2.$$

By Theorem 4.7 in [9],

$$\|f\|_{\mathcal{B}} \leq C \|f\|_X,$$

where  $\|f\|_X = \sup_{z \in B} |\nabla f(z)|(1 - |z|^2)$ . By the lemma in [5],

$$\|f\|_X \leq c \sup_{a \in B} \|f_a\|_{L_a^p}.$$

Therefore

$$(13) \quad \|f\|_{\mathcal{B}}^p \leq C J_2.$$

Because  $|\varphi_a(z)| < 1$  for  $z, a \in B$ , we know

$$|\varphi_a(z)|^{-2n+2} \leq |\varphi_a(z)|^{-2(n-\frac{1}{n})}.$$

By Lemma 1 and  $G(z, a) = g(\varphi_a(z))$ ,

$$(1 - |\varphi_a(z)|^2)^{n+1} |\varphi_a(z)|^{-2(n-\frac{1}{n})} \leq C(G(z, a))^{1+\frac{1}{n}}.$$

Hence

$$\begin{aligned} J_2 &\leq \sup_{a \in B} \int_B |\tilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} \cdot (1 - |\varphi_a(z)|^2)^{n+1} |\varphi_a(z)|^{-2(n-\frac{1}{n})} d\lambda(z) \\ (14) \quad &\leq C \sup_{a \in B} \int_B |\tilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} (G(z, a))^{1+\frac{1}{n}} d\lambda(z) \\ &= C J_3. \end{aligned}$$

Now let  $\|f\|_{\mathcal{B}} < \infty$ . By Theorem 4.7 in [9],

$$|\nabla f(z)|(1 - |z|^2) \leq C_1 \|f\|_{\mathcal{B}}.$$

Thus by Lemma 2.2 in [2],

$$|\nabla_T f(z)|(1 - |z|^2)^{\frac{1}{2}} \leq C_2 \|f\|_{\mathcal{B}},$$

where  $\nabla_T f$  is the complex tangential gradient of  $f$ . Hence by the proof of Theorem 2.4 in [2],

$$\begin{aligned} |\tilde{\nabla} f(z)|^2 &= \tilde{\Delta}|f|^2(z) \\ &\leq 4(1 - |z|^2)^2 |\nabla f(z)|^2 + 4(1 - |z|^2) |\nabla_T f(z)|^2 \\ &\leq C \|f\|_{\mathcal{B}}^2. \end{aligned}$$

From this and Lemma 1,

$$\begin{aligned}
 J_3(a) &= \int_B |\tilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} (G(z, a))^{1+\frac{1}{n}} d\lambda(z) \\
 (15) \quad &\leq C \|f\|_{\mathcal{D}}^2 \int_B |f \circ \varphi_a(w) - f \circ \varphi_a(0)|^{p-2} (g(w))^{1+\frac{1}{n}} \\
 &\quad \cdot (1 - |w|^2)^{-n-1} dm(w) \\
 &\leq C \|f\|_{\mathcal{D}}^2 \int_B |f_a(w)|^{p-2} |w|^{\frac{-2(n^2-1)}{n}} dm(w).
 \end{aligned}$$

When  $p = 2$

$$\begin{aligned}
 J_3(a) &\leq C \|f\|_{\mathcal{D}}^2 \int_B |w|^{\frac{-2(n^2-1)}{n}} dm(w) \\
 (16) \quad &= 2nC \|f\|_{\mathcal{D}}^2 \int_0^1 r^{2n-1-\frac{2(n^2-1)}{n}} dr \\
 &= 2nC \|f\|_{\mathcal{D}}^2 \int_0^1 r^{\frac{2}{n}-1} dr = n^2 C \|f\|_{\mathcal{D}}^2.
 \end{aligned}$$

When  $p > 2$ , let  $\alpha = \max(n^2 + 1, \frac{1}{p-2})$ ; then it is easy to know that

$$\left( \int_B |w|^{\frac{-2(n^2-1)}{n} \frac{\alpha}{\alpha-1}} dm(w) \right)^{1-\frac{1}{\alpha}} = M < \infty.$$

By the Lemma in [5],

$$\left( \int_B |f_a(w)|^{(p-2)\alpha} dm(w) \right)^{1/\alpha} \leq C(\Gamma((p-2)\alpha + 1))^{1/\alpha} \|f\|_{\mathcal{D}}^{p-2}.$$

Thus, by (15), using the Hölder inequality

$$\begin{aligned}
 J_3(a) &\leq C \|f\|_{\mathcal{D}}^2 \left( \int_B |f_a(w)|^{(p-2)\alpha} dm(w) \right)^{\frac{1}{\alpha}} \\
 (17) \quad &\quad \cdot \left( \int_B |w|^{\frac{-2(n^2-1)}{n} \frac{\alpha}{\alpha-1}} dm(w) \right)^{1-\frac{1}{\alpha}} \\
 &\leq CM(\Gamma((p-2)\alpha + 1))^{\frac{1}{\alpha}} \|f\|_{\mathcal{D}}^p.
 \end{aligned}$$

By (16) and (17), for  $p \geq 2$ ,

$$(18) \quad J_3 = \sup_{a \in B} J_3(a) \leq C(\Gamma((p-2)\alpha + 1))^{1/\alpha} \|f\|_{\mathcal{D}}^p.$$

By (13), (14) and (18), the quantities  $\|f\|_{\mathcal{D}}^p$ ,  $J_2$  and  $J_3$  are equivalent. The proof of Theorem 2 is complete.

*Remark 3.* It is authors' belief that the results of Theorem 2 should hold for all  $p$ 's, that is, also for  $0 < p < 2$ . In this case more delicate techniques seem to be needed.

*Proof of Theorem 3.* First, let  $f \in \mathcal{B}$ . For each integer  $k > 0$ , let  $\alpha = n^2 + 1$  in (18), then we get

$$\begin{aligned}
 I_{k+2}(a) &= \int_B |\tilde{\nabla} f(z)|^2 |f(z) - f(a)|^k (G(z, a))^{1+\frac{1}{n}} d\lambda(z) \\
 &\leq C(\Gamma(k(n^2 + 1) + 1))^{\frac{1}{n^2+1}} \|f\|_{\mathcal{B}}^{k+2}.
 \end{aligned}$$

It is easy to see that  $(\Gamma(k(n^2 + 1) + 1))^{\frac{1}{n^2+1}} \leq (n^2 + 1)^k k!$ . Hence

$$I_{k+2}(a) \leq C(n^2 + 1)^k k! \|f\|_{\mathcal{B}}^{k+2}.$$

Note that, when  $k = 0$ , the above inequality is also valid by (16). Taking a constant  $\tau$ ,  $0 < \tau < \frac{1}{n^2+1}$ , then if we set  $\beta = \tau/\|f\|_{\mathcal{B}}$ , we get

$$\begin{aligned} & e^{\beta t} \int_{E_{a,t}} |\tilde{\nabla} f(z)|^2 (G(z, a))^{1+\frac{1}{n}} d\lambda(z) \\ & \leq \int_{E_{a,t}} |\tilde{\nabla} f(z)|^2 e^{\beta|f(z)-f(a)|} (G(z, a))^{1+\frac{1}{n}} d\lambda(z) \\ & \leq \sum_{k=0}^{\infty} \frac{\beta^k}{k!} I_{k+2}(a) \leq C \sum_{k=0}^{\infty} \frac{\beta^k}{k!} (n^2 + 1)^k k! \|f\|_{\mathcal{B}}^{k+2} \\ & = C \|f\|_{\mathcal{B}}^2 \sum_{k=0}^{\infty} ((n^2 + 1)\tau)^k = K < \infty, \end{aligned}$$

where  $K = K_0 \|f\|_{\mathcal{B}}^2$ ,  $K_0$  is an absolute constant. Hence

$$(19) \quad \int_{E_{a,t}} |\tilde{\nabla} f(z)|^2 (G(z, a))^{1+\frac{1}{n}} d\lambda(z) \leq K e^{-\beta t}.$$

Conversely, let  $f$  satisfy (19); then

$$\int_0^{\infty} dt \int_{E_{a,t}} |\tilde{\nabla} f(z)|^2 (G(z, a))^{1+\frac{1}{n}} d\lambda(z) \leq K \int_0^{\infty} e^{-\beta t} dt = \frac{K}{\beta} < \infty.$$

But

$$\begin{aligned} & \int_0^{\infty} dt \int_{E_{a,t}} |\tilde{\nabla} f(z)|^2 (G(z, a))^{1+\frac{1}{n}} d\lambda(z) \\ & = \int_B |\tilde{\nabla} f(z)|^2 (G(z, a))^{1+\frac{1}{n}} \left( \int_0^{|f(z)-f(a)|} dt \right) d\lambda(z) \\ & = \int_B |\tilde{\nabla} f(z)|^2 |f(z) - f(a)| (G(z, a))^{1+\frac{1}{n}} d\lambda(z). \end{aligned}$$

So we get

$$\sup_{a \in B} \int_B |\tilde{\nabla} f(z)|^2 |f(z) - f(a)| (G(z, a))^{1+\frac{1}{n}} d\lambda(z) \leq \frac{K}{\beta} < \infty.$$

By Theorem 2 with  $p = 3$ , we know that  $f \in \mathcal{B}(B)$ . The proof is complete.

### 5. CHARACTERIZATIONS OF BMOA IN THE UNIT BALL

Let  $f \in H^1(B)$ , the Hardy space in the unit ball of  $\mathbb{C}^n$ . We say that  $f \in \text{BMOA}(\partial B)$  if its radial limit function  $f^*$  is a function of bounded mean oscillations on  $\partial B$  with respect to nonisotropic balls generated by the nonisotropic metric  $\rho(\zeta, \eta) = |1 - \langle \zeta, \eta \rangle|^{1/2}$  on  $\partial B$ . See [3] for details.

Let  $f_a = f \circ \varphi_a(\cdot) - f \circ \varphi_a(0)$ . In [4], Ouyang proved that a holomorphic function  $f \in \text{BMOA}(\partial B)$  if and only if

$$(20) \quad \sup_{a \in B} \|f_a\|_{H^p}^p < \infty.$$

Furthermore, he proved that if  $f \in \text{BMOA}(\partial B)$ , then

$$(21) \quad \sup_{a \in B} \|f_a\|_{H^p}^p \leq \frac{K\Gamma(p+1)}{C^p} \|f\|_{**}^p < \infty$$

where

$$\|f\|_{**} = \sup_{a \in B} \|f_a\|_{H^1}.$$

Now replacing  $f$  in (10) with  $f_a$ , we get

$$(22) \quad \begin{aligned} \|f_a\|_{H^p}^p &= \frac{p^2}{4} \int_B |\tilde{\nabla} f_a(w)|^2 |f \circ \varphi_a(w) - f \circ \varphi_a(0)|^{p-2} g(w) d\lambda(w) \\ &= \frac{p^2}{4} \int_B |\tilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} G(z, a) d\lambda(z). \end{aligned}$$

By (20), (21) and (22), we get the following

**Proposition 1.** For  $0 < p < \infty$ , a holomorphic function  $f \in \text{BMOA}(\partial B)$  if and only if

$$\sup_{a \in B} \int_B |\tilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} G(z, a) d\lambda(z) < \infty.$$

Moreover, if  $f \in \text{BMOA}(\partial B)$ , we have

$$(23) \quad \sup_{a \in B} \int_B |\tilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} G(z, a) d\lambda(z) \leq \frac{K\Gamma(p+1)}{C^p} \|f\|_{**}^p.$$

*Remark 4.* When  $p = 2$ , the above result was proved by J. S. Choa and B. R. Choe (see [1, Theorem A]).

Using (23) and a similar method of the proof of Theorem 3, we can obtain an exponential decay characterization of  $\text{BMOA}(\partial B)$  as follows.

**Theorem 4.** A holomorphic function  $f \in \text{BMOA}(\partial B)$  if and only if for every  $a \in B$  and every  $t > 0$ ,

$$\int_{E_{a,t}} |\tilde{\nabla} f(z)|^2 (G(z, a)) d\lambda(z) \leq Ke^{-\beta t}$$

where  $E_{a,t} = \{z \in B : |f(z) - f(a)| > t\}$ , and  $K, \beta > 0$  are constants. When  $f \in \text{BMOA}(\partial B)$ ,  $K = K_0 \|f\|_{**}^2$ ,  $\beta = C / \|f\|_{**}$ , where  $K_0$  and  $C$  are absolute constants.

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