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A VOLTERRA TYPE OPERATOR ON SPACES OF ANALYTIC FUNCTIONS

ARISTOMENIS G. SISKAKIS AND RUHAN ZHAO

Abstract. The main results are conditions on g such that the Volterra type operator

$$\mathsf{J}_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)\,d\zeta,$$

is bounded or compact on BMOA. We also point out certain information when J_g is considered as an operator on a general space X of analytic functions on the disc.

1. Introduction

Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}$. For g analytic on the disc consider the linear transformation

$$\mathsf{J}_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)\,d\zeta,$$

acting on functions f analytic on $\mathbb{D}$.

These operators arise naturally, with symbols g of a certain special form, in the study of semigroups of composition operators on spaces of analytic functions, see [Si, page 240] for details. When $g(z) = z$ or $g(z) = \log(1/(1-z))$, J_g is the integration operator or the Cesàro operator respectively.

It has been shown in [AleSi1] that J_g is a bounded (compact) operator on the Hardy spaces H^p, $1 \leq p < \infty$, if and only if $g \in$ BMOA ($g \in$ VMOA), the space of analytic functions whose boundary values have bounded (vanishing) mean oscillation, and further $\|\mathsf{J}_g\| \sim \|g\|_{BMOA}$. Analogous results on some general weighted Bergman spaces were shown in [AleSi2] with the Bloch and little Bloch space replacing BMOA and VMOA in the characterization of boundedness and compactness. Because of the connection with composition semigroups, these results can be applied to obtain properties of the resolvent operators of such semigroups.

Composition semigroups can be studied on other spaces of analytic functions and this leads to questions about the operator J_g on such spaces. The general problem may be stated as follows. Given a space $X = (X, \|\ \|)$ describe those symbols g for which J_g is bounded or compact or has some other operator theoretic property as an operator on X.

This article consists of two parts. In the first part we describe some facts for J_g acting on a general Banach or Hilbert space of analytic functions. In the second part we characterize those g which give bounded or compact operators when $X =$BMOA.

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2. Some general properties

Let X be a Banach space of analytic functions on $\mathbb{D}$ and define the spaces of functions V and V_0 as follows:

$$V = V_X = \{g \text{ analytic on } \mathbb{D} : \mathsf{J}_g : X \to X \text{ is bounded}\}$$
$$V_0 = V_{0,X} = \{g \text{ analytic on } \mathbb{D} : \mathsf{J}_g : X \to X \text{ is compact}\}.$$

Clearly $V_0 \subseteq V$ and both contain the constant functions. Because $\mathsf{J}_{\lambda g} = \lambda \mathsf{J}_g$ and $\mathsf{J}_{g+h} = \mathsf{J}_g + \mathsf{J}_h$, both are vector spaces. We introduce the norm

$$\|g\| = \|g\|_V = |g(0)| + \|\mathsf{J}_g\|_{X \to X}.$$

Proposition 2.1. *Suppose that convergence in the norm of X implies uniform convergence on compact subsets of $\mathbb{D}$. Then V and V_0 are complete and are therefore Banach spaces.*

Proof. Let $\{g_n\}$ be a Cauchy sequence in V. Then $\{g_n(0)\}$ is a Cauchy sequence of scalars and $\{\mathsf{J}_{g_n}\}$ is a Cauchy sequence in the space $L(X)$ of all bounded linear operators on X. Thus there is $c_0 \in \mathbb{C}$ and $T \in L(X)$ such that

$$\lim_{n\to\infty} g_n(0) = c_0 \qquad \text{and} \qquad \lim_{n\to\infty} \|\mathsf{J}_{g_n} - T\| = 0.$$

Pick $\phi \in X$ not identically 0 (if X contains the constants we can pick $\phi \equiv 1$) and let $\Phi(z) = T(\phi)(z)$, then

$$\lim_{n\to\infty} \mathsf{J}_{g_n}(\phi) = \Phi,$$

in the norm of X. It follows from the hypothesis that the sequence $\{\mathsf{J}_{g_n}(\phi)\}$ converges to Φ uniformly on compact subsets of the disc and the same is true for the sequence of derivatives hence,

$$\phi(z) g_n'(z) \longrightarrow \Phi'(z)$$

uniformly on compact sets. From the convergence it is clear that Φ' vanishes at all zeros of ϕ with multiplicity counted thus $\Phi'(z)/\phi(z)$ is analytic on the disc and

$$g_n'(z) \longrightarrow \frac{\Phi'(z)}{\phi(z)},$$

for each $z \in \mathbb{D}$. Let g be defined by

$$g(z) = c_0 + \int_0^z \frac{\Phi'(\zeta)}{\phi(\zeta)}\, d\zeta.$$

We will show $T = \mathsf{J}_g$ which means $g \in V$ and $\lim_{n\to\infty} g_n = g$ in the norm of V. Let $f \in X$ be arbitrary and write F=T(f). Arguing as above on the implications of convergence in X we find $F'(z)$ as a limit

$$F'(z) = \lim_{n\to\infty} f(z) g_n'(z) = f(z) \lim_{n\to\infty} g_n'(z) = f(z)\frac{\Phi'(z)}{\phi(z)} = f(z) g'(z),$$

for each $z \in \mathbb{D}$. Integrating and noticing that $\lim \mathsf{J}_{g_n}(f)(0) = 0$, so that $F(0) = 0$, we obtain

$$T(f)(z) = F(z) = \int_0^z f(\zeta) g'(\zeta)\, d\zeta = \mathsf{J}_g(f)(z),$$

for each $f \in X$, i.e. $T = \mathsf{J}_g$ as desired. Thus V is a Banach space.

The argument for V_0 is the same. If $g_n \in V_0$ then J_{g_n} are compact and so is their limit T. Thus the function g in the above argument is in V_0 and we see that V_0 is a closed subspace of V. $\square$

Proposition 2.2. *Suppose the multiplication operator* $M_z : f(z) \to zf(z)$ *is bounded on* X *then:*
(i) If the integration operator $\mathsf{J}_z(f)(z) = \int_0^z f(\zeta)\, d\zeta$ *is bounded on* X *then* V *contains all polynomials.*
(ii) If the integration operator J_z *is compact on* X *then* V_0 *contains all polynomials.*

Proof. Let $f \in X$ and $n \geq 1$ then

$$\begin{aligned} \mathsf{J}_{z^n}(f)(z) &= n \int_0^z f(\zeta)\zeta^{n-1}\, d\zeta \\ &= n \int_0^z M_z^{n-1}(f)(\zeta)\, d\zeta \\ &= n\mathsf{J}_z \circ M_z^{n-1}(f)(z), \end{aligned}$$

i.e. $\mathsf{J}_{z^n} = n\mathsf{J}_z \circ M_z^{n-1}$. It follows that $z^n \in V$ (or $z^n \in V_0$) whenever $z \in V$ (or $z \in V_0$). Since V and V_0 are vector spaces the assertions follow. □

Here and later we will use the Möbius automorphisms,

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z}, \qquad a \in \mathbb{D}.$$

which map the disc conformally onto itself and exchange a with 0. We denote by C_a the composition operator

$$C_a(f) = f \circ \phi_a,$$

defined on functions f analytic on $\mathbb{D}$. Note that $\phi_a \circ \phi_a(z) = z$ so ϕ_a is its own inverse. It follows that C_a is also its own inverse.

Proposition 2.3. *Suppose point evaluations* $L_\lambda(f) = f(\lambda)$ *are bounded on* X *for* λ *in the disc and that the composition operator* C_a *is bounded on* X*. Then* V *and* V_0 *are preserved under composition with* ϕ_a*.*

Proof. Let $f \in X$ and write $F(z) = \mathsf{J}_g(f)(z)$ then $F'(z) = f(z)g'(z)$. Compose with ϕ_a and multiply both sides by ϕ_a' to obtain

$$(F \circ \phi_a)'(z) = (f \circ \phi_a)(z)(g \circ \phi_a)'(z),$$

thus

$$(F \circ \phi_a)(z) - (F \circ \phi_a)(0) = \int_0^z (f \circ \phi_a)(\zeta)(g \circ \phi_a)'(\zeta)\, d\zeta.$$

We write this equation in terms of C_a which by hypothesis is bounded on X,

$$C_a \circ \mathsf{J}_g(f)(z) - C_a \circ \mathsf{J}_g(f)(0) = \mathsf{J}_{g\circ\phi_a} \circ C_a(f),$$

or equivalently,

$$I_0 \circ C_a \circ \mathsf{J}_g \circ C_a = \mathsf{J}_{g\circ\phi_a},$$

where I_0 is the operator $I_0(f)(z) = f(z) - f(0)$, which is bounded on X by the hypothesis on point evaluations. It follows that $g \in V$ (or $g \in V_0$) if and only if $g \circ \phi_a \in V$ (resp. $g \circ \phi_a \in V_0$). □

Proposition 2.4. *Suppose* X *contains the constant functions. Then* $V \subseteq X$ *and there is a constant* C *such that* $\|g\|_X \leq C\|g\|_V$ *for each* $g \in V$*.*

Proof. Suppose $g \in V$. For the constant function $1 \in X$ we have

$$\mathsf{J}_g(1)(z) = \int_0^z g'(\zeta)\, d\zeta = g(z) - g(0) \in X$$

so $g \in X$. Further,

$$\begin{aligned} \|g\|_X = \|g(0) + g - g(0)\|_X &\leq \|g(0)\|_X + \|g(z) - g(0)\|_X \\ &= |g(0)|\|1\|_X + \|\mathsf{J}_g(1)\|_X \\ &\leq |g(0)|\|1\|_X + \|\mathsf{J}_g\|_{X\to X}\|1\|_X \\ &= C\|g\|_V \end{aligned}$$

with $C = \|1\|_X$. □

When X is a Hardy or a Bergman space the identification of the spaces V and V_0 was obtained in [AleSi1] and [AleSi2]. If $X = H^p$, $1 \leq p < \infty$, then $V = BMOA$ and $V_0 = VMOA$. For $X = A^p$, $1 \leq p < \infty$, the usual Bergman space, the spaces are $V = \mathcal{B}$ the Bloch space, and $V_0 = \mathcal{B}_0$ the little Bloch space.

When the underlying space is a Hilbert space, additional classes of g can be singled out by requiring J_g to belong to some operator ideal (this of course can be done on Banach spaces but we stay with the more familiar situation). We describe briefly how these classes come out.

Recall that if H is a separable Hilbert space then for $1 \leq p < \infty$ the Schatten class $\mathcal{S}^p = \mathcal{S}^p(H)$ is the set of all bounded linear operators $T : H \to H$ for which the sequence $\{s_k\}$ of singular numbers $s_k = s_k(T) = \inf\{\|T - F\| : rankF < k\}$ is in the sequence space l^p. $\mathcal{S}^p$ is a two sided ideal in the space of all bounded operators on H. The Schatten norm of T is defined by $\|T\|_{\mathcal{S}^p} = \|\{s_k\}\|_{l^p}$ and with this norm $\mathcal{S}^p$ is a Banach space. The Hilbert-Schmidt class $\mathcal{S}^2$ is a Hilbert space with inner product $\langle T, S\rangle_{\mathcal{S}^2} = trace(TS)$. Because the first singular value s_1 is just the operator norm $\|T\|_{H\to H}$ we see that $\|T\|_{H\to H} \leq \|T\|_{\mathcal{S}^p}$. It follows that any sequence of operators converging in the norm $\|\ \|_{\mathcal{S}^p}$ will also converge in the operator norm to the same limit.

Let H be a Hilbert space consisting of analytic functions on $\mathbb{D}$. For each $p \geq 1$ define

$$V_p = V_{p,H} = \{g \text{ analytic on } \mathbb{D} : \mathsf{J}_g \in \mathcal{S}^p\}.$$

Clearly V_p is a vector space always containing the constant functions. We can give it a norm

$$\|g\|_{V_p} = |g(0)| + \|\mathsf{J}_g\|_{\mathcal{S}^p},$$

under which V_p is a Banach space. This follows from the fact that a Cauchy sequence of operators in the $\mathcal{S}^p$-norm is also Cauchy in the operator norm so the proof of Proposition (2.1) applies. For $p = 2$ the inner product

$$\langle g, h\rangle = g(0)\overline{h(0)} + \langle \mathsf{J}_g, \mathsf{J}_h\rangle_{\mathcal{S}^2}$$

makes V_2 a Hilbert space.

Further if point evaluations are bounded linear functionals on H and the compositions $C_a(f) = f \circ \phi_a$ with Möbius automorphisms of $\mathbb{D}$ are bounded operators on H then the proof of Proposition (2.3) shows that V_p are preserved by composition with ϕ_a.

When $H = H^2$ the resulting classes are $V_p = B_p$ the Besov spaces for $1 < p < \infty$. For $p = 1$ the space V_1 contains only constants. The same Besov spaces are obtained when J_g acts on the Bergman space A^2. Details can be found in the references

[AleSi1],[AleSi2]. As a further example let $H = \mathcal{D}$, the Dirichlet space of analytic functions on $\mathbb{D}$ that have square integrable derivatives. It is easy to see that J_g acting on $\mathcal{D}$ is Hilbert-Schmidt if and only if g satisfies

$$\int_{\mathbb{D}} |g'(z)|^2 \log \frac{1}{1-|z|^2}\, dm(z) < \infty.$$

The space of g 's obtained is a weighted Dirichlet space with logarithmic weight and is properly contained in $\mathcal{D}$.

3. J_g ON BMOA

When $X =$ BMOA we find a Carleson type condition on g for J_g to be bounded or compact. We first give some background about Carleson measures and the space BMOA.

Let $d\theta/2\pi$ and $dm(z) = r dr d\theta$ denote the Lebesgue measure on $\partial\mathbb{D}$ and $\mathbb{D}$ respectively. Recall that for an arc I in $\partial\mathbb{D}$ the Carleson box based on I is

$$S(I) = \{z : 1 - |I| \leq |z| < 1,\ z/|z| \in I\}$$

where $|I|$ is the Lebesgue length of the arc. A positive measure μ on the disc is a Carleson measure if

$$\mathcal{N}(\mu) = \sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|} < \infty, \tag{3.1}$$

and μ is a vanishing Carleson measure if $\mu(S(I)) = o(|I|)$ as $|I| \to 0$. The condition (3.1) characterizes the measures for which

$$\int_{\mathbb{D}} |f(z)|^2\, d\mu(z) \leq C \int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta, \qquad f \in H^2, \tag{3.2}$$

or equivalently the inclusion map $\mathcal{I} : H^2 \hookrightarrow L^2(\mathbb{D}, \mu)$ which embeds the Hardy space into the Lebesgue space of μ is bounded. If this is the case $\mathcal{N}(\mu)$ is comparable to $\|\mathcal{I}\|^2_{H^2 \to L^2(\mathbb{D},d\mu)}$. Vanishing Carleson measures are those for which this inclusion is compact.

The space BMOA consists of those functions $f \in H^2$ for which the set of Möbius translates $\{f \circ \phi_a(z) - f(a) : a \in \mathbb{D}\}$ is bounded in H^2, i.e.,

$$\rho(f) = \sup_{a \in \mathbb{D}} \|f \circ \phi_a(z) - f(a)\|_{H^2} \sim \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 (1 - |\phi_a(z)|^2)\, dm(z) \right)^{1/2} < \infty,$$

and VMOA contains those f for which

$$\lim_{|a| \to 1} \|f \circ \phi_a(z) - f(a)\|_{H^2} = 0.$$

BMOA is a Banach space under the norm $\|f\|_* = |f(0)| + \rho(f)$ and VMOA is a closed subspace and coincides with the closure of the polynomials under this norm. Clearly $\rho(f \circ \phi_a) = \rho(f)$ for each $f \in BMOA$ and each $a \in \mathbb{D}$.

There is a close relation of BMOA to Carleson measures. A function g is in BMOA if and only if the measure

$$d\mu_g(z) = |g'(z)|^2 (1 - |z|^2)\, dm(z)$$

is Carleson and $g \in$ VMOA if and only if this measure is vanishing Carleson. In that case the Carleson norm $\mathcal{N}(\mu_g)$ is comparable to $\rho(g)^2$. Details on these and other information on BMOA can be found in the books [Gar] or [Zhu].

We now return to the operator J_g to consider its action on BMOA. It is easy to see that membership of g in BMOA, which characterizes boundedness of J_g on Hardy spaces, is not sufficient to make J_g bounded on BMOA. For example let $g(z) = \log \frac{1}{1-z} \in$ BMOA then

$$\mathsf{J}_g(g)(z) = \frac{1}{2} \log^2 \frac{1}{1-z},$$

which violates the growth inequality (3.4) of BMOA functions. The following theorem says that the correct condition is bounded logarithmic mean oscillation.

Theorem 3.1. *The operator J_g is bounded on BMOA if and only if*

$$\|g\|^2_{LMOA} = \sup_{I \subset \partial\mathbb{D}} \left\{ \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{S(I)} |g'(z)|^2 (1-|z|^2)\, dm(z) \right\} < \infty, \tag{3.3}$$

and in that case $\|\mathsf{J}_g\|$ is comparable to $\|g\|_{LMOA}$.

Proof. Suppose the condition holds and let $f \in$ BMOA. We will show $F = \mathsf{J}_g(f) \in$ BMOA by showing that

$$d\mu_F = |F'(z)|^2 (1-|z|^2)\, dm(z)$$

is a Carleson measure. Let $I \subset \partial\mathbb{D}$ be an arc then

$$\begin{aligned}
\frac{1}{|I|} \int_{S(I)} |F'(z)|^2 (1-|z|^2)\, dm(z) &= \frac{1}{|I|} \int_{S(I)} |f(z)|^2 |g'(z)|^2 (1-|z|^2)\, dm(z) \\
&\le \frac{2}{|I|} \int_{S(I)} |f(z)-f(u)|^2 |g'(z)|^2 (1-|z|^2)\, dm(z) \\
&+ \frac{2}{|I|} \int_{S(I)} |f(u)|^2 |g'(z)|^2 (1-|z|^2)\, dm(z), \\
&= 2J_1 + 2J_2,
\end{aligned}$$

where we used the scalar inequality $|x+y|^2 \le 2|x|^2 + 2|y|^2$ and u is a point in the disc to be specified in the next step. To estimate J_1 let $\xi \in \partial\mathbb{D}$ be the center of I and consider the disc automorphism $\phi_u(z) = \frac{u-z}{1-\bar{u}z}$ where $u = (1-|I|)\xi \in \mathbb{D}$. By a simple geometric argument there is an absolute constant $C > 0$ such that

$$\frac{1-|u|^2}{|1-\bar{u}z|^2} \ge \frac{C}{|I|}, \qquad z \in S(I),$$

so that $1 - |\phi_u(z)|^2 \ge C \frac{1-|z|^2}{|I|}$ for all $z \in S(I)$. It follows that

$$\begin{aligned}
J_1 &\le \frac{1}{C} \int_{S(I)} |f(z)-f(u)|^2 |g'(z)|^2 (1-|\phi_u(z)|^2)\, dm(z) \\
&\le C' \int_{\mathbb{D}} |f(z)-f(u)|^2 |g'(z)|^2 (1-|\phi_u(z)|^2)\, dm(z) \\
&= C' \int_{\mathbb{D}} |f\circ\phi_u(w) - f(u)|^2 |(g\circ\phi_u)'(w)|^2 (1-|w|^2)\, dm(w),
\end{aligned}$$

with a change of variable in the last step. Now condition (3.3) clearly implies g belongs to BMOA and so does $g \circ \phi_u$. Thus $d\mu_g = |(g \circ \phi_u)'(w)|^2(1-|w|^2)\, dm(w)$ is a Carleson measure and we can apply (3.2) to obtain

$$\begin{aligned}
&\le C'\rho(g\circ\phi_u)^2 \int_0^{2\pi} |f\circ\phi_u(e^{i\theta}) - f(u)|^2\, d\theta \\
&\le C'\rho(g)^2\rho(f)^2 \\
&\le C'\|g\|_\star^2\|f\|_\star^2.
\end{aligned}$$

Next the estimate for J_2 is a simple application of the growth inequality for BMOA functions f,

$$|f(z)| \le C\|f\|_\star \log\frac{2}{1-|z|}, \qquad z \in \mathbb{D}, \tag{3.4}$$

where C does not depend on f. We obtain,

$$\begin{aligned}
J_2 &= \frac{|f(u)|^2}{|I|}\int_{S(I)} |g'(z)|^2(1-|z|^2)\, dm(z) \\
&\le C^2\|f\|_\star^2 \frac{\log^2\frac{2}{1-|u|}}{|I|}\int_{S(I)} |g'(z)|^2(1-|z|^2)\, dm(z) \\
&\le C''\|g\|_{LMOA}^2\|f\|_\star^2.
\end{aligned}$$

Putting together the two inequalities for J_1 and J_2 we conclude that $d\mu_F$ is a Carleson measure and

$$\begin{aligned}
\|\mathsf{J}_g(f)\|_\star^2 = \rho(F)^2 &\le C\mathcal{N}(\mu_F) \\
&\le C(C'\|g\|_\star^2 + C''\|g\|_{LMOA}^2)\|f\|_\star^2 \\
&\le C'''\|g\|_{LMOA}^2\|f\|_\star^2,
\end{aligned}$$

because clearly $\|g\|_\star \le C\|g\|_{LMOA}$.

Conversely suppose J_g : BMOA $\to$ BMOA is bounded. We will use the test functions

$$f_a(z) = \log\frac{1}{1-\bar a z}, \qquad a \in \mathbb{D},$$

which form a bounded set in BMOA, in fact $\|f_a\|_\star \le \|\log\frac{1}{1-z}\|_\star$. For an arc $I \subset \partial\mathbb{D}$ let $a = (1-|I|)\xi$ with ξ the center of I. Then there is a constant C such that

$$\frac{1}{C}\log\frac{2}{|I|} \le |f_a(z)| \le C\log\frac{2}{|I|}$$

for all $z \in S(I)$ and this gives

$$\begin{aligned}
\frac{\log^2\frac{2}{|I|}}{|I|}\int_{S(I)} |g'(z)|^2(1-|z|^2)\, dm(z) &\le \frac{C^2}{|I|}\int_{S(I)} |f_a(z)|^2|g'(z)|^2(1-|z|^2)\, dm(z) \\
&= \frac{C^2}{|I|}\int_{S(I)} |(\mathsf{J}_g f_a)'(z)|^2(1-|z|^2)\, dm(z) \\
&\le C'\|\mathsf{J}_g(f_a)\|_\star^2 \\
&\le C'\|\mathsf{J}_g\|^2\|f_a\|_\star^2 \\
&\le C''\|\mathsf{J}_g\|^2.
\end{aligned}$$

It follows that (3.3) holds and $\|g\|_{LMOA} \le C\|\mathsf{J}_g\|$. This finishes the proof.

□

Proposition 3.2. *Suppose* $\mathsf{J}_g : BMOA \to BMOA$ *is a bounded operator. Then* $\mathsf{J}_g(VMOA) \subset VMOA$.

Proof. If J_g is bounded on BMOA then (3.3) holds. Now notice that if g satisfies (3.3) then in fact $g \in$ VMOA. Also $\mathsf{J}_g(1) = g - g(0)$ thus the constant functions are mapped into VMOA. If n is a positive integer then an integration by parts gives

$$\mathsf{J}_g(z^n) = z^n g(z) - n \int_0^z \zeta^{n-1} g(\zeta)\, d\zeta,$$

and because the multiplication by z and the integration operator are bounded on VMOA we see that $\mathsf{J}_g(z^n) \in$ VMOA. It follows that $\mathsf{J}_g(p) \in$ VMOA for each polynomial p. Next let $f \in$ VMOA be arbitrary. There is a sequence $\{p_n\}$ of polynomials such that $\|f - p_n\|_\star \to 0$ and we have

$$\|\mathsf{J}_g(f) - \mathsf{J}_g(p_n)\|_\star = \|\mathsf{J}_g(f - p_n)\|_\star \le \|\mathsf{J}_g\| \|f - p_n\|_\star.$$

This shows that $\mathsf{J}_g(f)$ can be approximated in the $\| \, \|_\star$ norm by VMOA functions. Since VMOA is closed in this norm the assertion follows.

□

An important property of the spaces BMOA and VMOA is their duality relation with the Hardy space H^1. The pairing

$$\langle f, h \rangle = \int_0^{2\pi} f(e^{i\theta}) \overline{h(e^{i\theta})} \, \frac{d\theta}{2\pi}$$

for $f \in$ VMOA and $h \in H^1$ identifies the dual space of VMOA as VMOA$^* \simeq H^1$ where $\simeq$ means Banach space isomorphism. The same pairing for $f \in H^1$ and $h \in$ BMOA gives the duality $(H^1)^* =$ BMOA. In both cases for a given h in one space the integral is defined only on a dense subset of functions f of the other.

Let g be a symbol such that J_g is bounded on VMOA and let $\mathsf{A}_g = \mathsf{J}_g^*$ be the adjoint operator acting on H^1. Let also A_g^* be the adjoint of A_g acting on BMOA. Then

$$\langle \mathsf{J}_g(f), h \rangle = \langle f, \mathsf{A}_g(h) \rangle = \overline{\langle \mathsf{A}_g(h), f \rangle} = \overline{\langle h, \mathsf{A}_g^*(f) \rangle} = \langle \mathsf{A}_g^*(f), h \rangle$$

for all $f \in$ VMOA and $h \in H^1$. Because VMOA is weak* dense in BMOA this says that $\mathsf{A}_g^* = \mathsf{J}_g$ as operators on BMOA. Thus J_g is bounded on BMOA. Together with Proposition (3.2) this proves the following

Corollary 3.3. *The following are equivalent*

(i) J_g *is bounded on BMOA.*
(ii) J_g *is bounded on VMOA.*
(iii) $\|g\|_{LMOA} < \infty$.

We now discuss two lemmas about the space $LMOA$ and its little "oh" version which will be needed later.

Lemma 3.4. *The seminorm* $\|g\|_{LMOA}$ *defined in (3.3) is equivalent to the seminorm given by*

$$(\|g\|'_{LMOA})^2 = \sup_{a \in \mathbb{D}} \left\{ (\log \frac{1}{1-|a|})^2 \int_{\mathbb{D}} |g'(z)|^2 (1 - |\phi_a(z)|^2) \, dm(z) \right\}.$$

Proof. We include the proof for completeness. With I, ξ and $a = (1-|I|)\xi$ as in Theorem (3.1) we have

$$\begin{aligned}\frac{\log^2 \frac{2}{|I|}}{|I|} \int_{S(I)} |g'(z)|^2(1-|z|^2)\, dm(z) \\ \leq & C \log^2 \frac{2}{1-|a|} \int_{S(I)} |g'(z)|^2(1-|\phi_a(z)|^2)\, dm(z) \\ \leq & C \log^2 \frac{2}{1-|a|} \int_{\mathbb{D}} |g'(z)|^2(1-|\phi_a(z)|^2)\, dm(z)\end{aligned}$$

where we have used the inequality $\frac{1-|z|^2}{|I|} \leq C(1-|\phi_a(z)|^2)$ for $z \in S(I)$. Taking sup the one direction follows.

For the converse we use a standard argument [Gar, page 239], see also [OF, Lemma 2.12]. If $|a| \leq \frac{3}{4}$ the estimate is trivial. Assume $|a| \geq \frac{3}{4}$ and let

$$E_n = \{z \in \mathbb{D} : |z - \frac{a}{|a|}| < 2^{n-1}(1-|a|)\}, \qquad n = 1, 2, ...N$$

where $N = N(a)$ is the smallest integer such that $2^{N-1}(1-|a|) \geq \frac{1}{\pi}$, so that $N \sim \log(\frac{1}{1-|a|})/\log 2$. An easy geometric argument shows that there is a constant C such that

$$\frac{1-|a|^2}{|1-\bar{a}z|^2} \leq \frac{C}{2^{2n}(1-|a|)}, \qquad z \in E_n \setminus E_{n-1}, \qquad n \geq 1,$$

(we set $E_0 = \emptyset$). Also for each n there is a Carleson box $S(I)$ with $|I| \sim 2^{n-1}(1-|a|)$ such that $E_n \subset S(I)$. Then we have

$$\begin{aligned}\int_{\mathbb{D}} |g'(z)|^2(1-|\phi_a(z)|^2)\, dm(z) =& \sum_{n=1}^{N} \int_{E_n \setminus E_{n-1}} |g'(z)|^2(1-|\phi_a(z)|^2)\, dm(z) \\ \leq & \sum_{n=1}^{N} \frac{C}{2^{2n}(1-|a|)} \int_{E_n \setminus E_{n-1}} |g'(z)|^2(1-|z|^2)\, dm(z) \\ \leq & \frac{C}{2} \left(\sum_{n=1}^{N} \frac{1}{2^n} \frac{1}{\log^2 \frac{2}{2^{n-1}(1-|a|)}} \right) \|g\|^2_{LMOA} \\ \leq & \frac{C'}{\log^2 \frac{2}{1-|a|}} \|g\|^2_{LMOA},\end{aligned}$$

and the conclusion follows.

□

Let $LMOA_0$ be the space of functions in $LMOA$ such that

$$\lim_{|I| \to 0} \left\{ \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{S(I)} |g'(z)|^2(1-|z|^2)\, dm(z) \right\} = 0,$$

or equivalently

$$\lim_{|a| \to 1} \left\{ (\log \frac{1}{1-|a|})^2 \int_{\mathbb{D}} |g'(z)|^2(1-|\phi_a(z)|^2)\, dm(z) \right\} = 0.$$

This space $LMOA_0$ sits inside $LMOA$ in the same way that $VMOA$ is inside BMOA. In particular we have,

Lemma 3.5. *Let* $g \in LMOA$ *and denote* $g_r(z) = g(rz)$, $0 < r < 1$. *Then* $g \in LMOA_0$ *if and only if* $\lim_{r\to 1} \|g - g_r\|_{LMOA} = 0$

Proof. Suppose $g \in LMOA$. From the Poisson formula

$$g_r(z) = \int_0^{2\pi} g(ze^{i\theta}) \frac{1 - r^2}{|e^{i\theta} - r|^2} \frac{d\theta}{2\pi},$$

we have

$$\begin{aligned}
&\log^2 \frac{1}{1-|a|} \int_{\mathbb{D}} |g_r'(z)|^2 (1 - |\phi_a(z)|^2)\, dm(z) \\
&\qquad \le \log^2 \frac{1}{1-|a|} \int_{\mathbb{D}} \int_0^{2\pi} |g(e^{i\theta} z)|^2 \frac{1 - r^2}{|e^{i\theta} - r|^2} \frac{d\theta}{2\pi} (1 - |\phi_a(z)|^2)\, dm(z) \\
&\qquad = \int_0^{2\pi} \left(\log^2 \frac{1}{1-|a|} \int_{\mathbb{D}} |g(e^{i\theta} z)|^2 (1 - |\phi_a(z)|^2)\, dm(z) \right) \frac{1 - r^2}{|e^{i\theta} - r|^2} \frac{d\theta}{2\pi}.
\end{aligned}$$

Since the inside parenthesis is bounded by $\|g\|^2_{LMOA}$ we can use the Lebesgue dominated convergence theorem to obtain

$$\lim_{|a|\to 1} \log^2 \frac{1}{1-|a|} \int_{\mathbb{D}} |g_r'(z)|^2 (1 - |\phi_a(z)|^2) dm(z) = 0$$

uniformly for $r \in (0, 1)$. Combining this with the fact that $g \in LMOA_0$ we see that for a given $\varepsilon > 0$ there is a $\delta \in (0, 1)$ such that

$$\sup_{|a|>\delta} \log^2 \frac{1}{1-|a|} \int_{\mathbb{D}} |g'(z) - g_r'(z)|^2 (1 - |\phi_a(z)|^2)\, dm(z) < \varepsilon \tag{3.5}$$

for all $r \in (0, 1)$. Next for $|a| \le \delta$,

$$\begin{aligned}
&\sup_{|a|\le\delta} \log^2 \frac{1}{1-|a|} \int_{\mathbb{D}} |g'(z) - g_r'(z)|^2 (1 - |\phi_a(z)|^2)\, dm(z) \\
&\qquad \le \log^2 \frac{1}{1-\delta} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g'(z) - g_r'(z)|^2 (1 - |\phi_a(z)|^2)\, dm(z) \\
&\qquad = \log^2 \frac{1}{1-\delta} \|g - g_r\|^2_*.
\end{aligned}$$

Now g is in particular in $VMOA$ and it is well known that the analogue of the lemma we are proving holds for functions in $VMOA$ with the $BMOA$ norm, i.e. $\lim_{r\to 1} \|g - g_r\|_* = 0$. Putting together this and (3.5) we have the desired conclusion.

Conversely suppose $g \in LMOA$ and $\lim_{r\to 1} \|g_r - g\|_{LMOA} = 0$. Then for $\varepsilon > 0$ there is an $r_0 \in (0, 1)$ such that for $r \in (r_0, 1)$, we have $\|g_r - g\|_{LMOA} < \varepsilon$. It follows that

$$\begin{aligned}
&\log^2 \frac{1}{1-|a|} \int_{\mathbb{D}} |g'(z)|^2 (1 - |\phi_a(z)|^2)\, dm(z) \\
&\qquad \le \log^2 \frac{1}{1-|a|} \int_{\mathbb{D}} (2|g'(z) - g_r'(z)|^2 + 2|g_r'(z)|^2)(1 - |\phi_a(z)|^2)\, dm(z) \\
&\qquad \le 2\varepsilon + 2 \log^2 \frac{1}{1-|a|} \int_{\mathbb{D}} |g_r'(z)|^2 (1 - |\phi_a(z)|^2)\, dm(z)
\end{aligned}$$

Now take $|a| \to 1$ then the last integral goes to zero because $g_r \in LMOA_0$ and the conclusion follows.

$\square$

For $g \in LMOA_0$ and a sequence $r_n \nearrow 1$ we can find a polynomial p_n such that $|g'_{r_n}(z) - p'_n(z)|^2 < 1/n$ for all $z \in \mathbb{D}$, hence $\|g_{r_n} - p_n\|^2_{LMOA} < \frac{C}{n}$. In particular the above lemma says that $LMOA_0$ is the closure of the polynomials in the $LMOA$ norm.

We now turn to compactness of J_g. Recall that if X is a Banach space and $T : X \to X$ a linear operator, T is said to be *compact* if for every bounded sequence $\{x_n\}$ in X, $\{T(x_n)\}$ has a convergent subsequence. T is *weakly compact* if for every bounded sequence $\{x_n\}$ in X, $\{T(x_n)\}$ has a weakly convergent subsequence. Every compact operator is weakly compact and in reflexive spaces every bounded operator is weakly compact. In our case, a useful characterization of weak compactness is [DuSch, page 482]: T is weakly compact if and only if $T^{**}(X^{**}) \subset X$ where T^{**} is the second adjoint of T and X is identified with its image under the natural embedding into its second dual X^{**}.

Theorem 3.6. *J_g is compact on BMOA if and only if $g \in LMOA_0$, i.e.*

$$\lim_{|I|\to 0} \left\{ \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{S(I)} |g'(z)|^2 (1 - |z|^2)\, dm(z) \right\} = 0. \tag{3.6}$$

Proof. Before we start we observe that because of (3.4) the unit ball of BMOA is a normal family of analytic functions. A normal family arguments shows J_g is compact on BMOA if and only if every sequence $\{f_n\}$ in BMOA with $\|f_n\|_\star \le 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ has a subsequence $\{f_{n_k}\}$ such that $\|\mathsf{J}_g(f_{n_k})\|_\star \to 0$.

Suppose first that J_g is compact on BMOA. We will show that every sequence of intervals whose length goes to zero has a subsequence such that (3.6) holds when the limit is taken over that subsequence. This clearly implies (3.6).

Let I_n be intervals with $|I_n| \to 0$ and let $u_n = (1 - |I_n|)\xi_n$ with ξ_n the center of I_n. The sequence $\{u_n\}$ has an accumulation point $u \in \partial\mathbb{D}$ and passing to a subsequence we may assume $\lim_{n\to\infty} u_n = u$. Let f_n, f_0 and q_n be the functions

$$f_n(z) = \log \frac{1}{1 - \bar{u}_n z}, \qquad f_0(z) = \log \frac{1}{1 - \bar{u} z}, \qquad q_n(z) = \log \frac{1 - \bar{u} z}{1 - \bar{u}_n z}.$$

As in the second part of the proof of Theorem (3.1) we have for each n,

$$\begin{aligned} &\frac{\left(\log \frac{2}{|I_n|}\right)^2}{|I_n|} \int_{S(I_n)} |g'(z)|^2 (1 - |z|^2)\, dm(z) \\ &\qquad \le \frac{C}{|I_n|} \int_{S(I_n)} |f_n(z)|^2 |g'(z)|^2 (1 - |z|^2)\, dm(z) \\ &\qquad \le \frac{2C}{|I_n|} \int_{S(I_n)} |q_n(z)|^2 |g'(z)|^2 (1 - |z|^2)\, dm(z) \\ &\qquad + \frac{2C}{|I_n|} \int_{S(I_n)} |f_0(z)|^2 |g'(z)|^2 (1 - |z|^2)\, dm(z) \\ &\qquad \le C' \|\mathsf{J}_g(q_n)\|_\star^2 + \frac{2C}{|I_n|} \int_{S(I_n)} |\mathsf{J}_g(f_0)'(z)|^2 (1 - |z|^2)\, dm(z). \end{aligned}$$

Now $\|q_n\|_\star \le 2\|\log\frac{1}{1-z}\|_\star$ and $q_n \to 0$ uniformly on compact sets so the compactness of J_g implies $\lim_{n\to\infty}\|\mathsf{J}_g(q_n)\|_\star = 0$. We show that the other term in the right hand side goes to zero. Because VMOA is a subspace of BMOA and they share the same norm, compactness of J_g on BMOA implies its compactness on VMOA. Thus J_g is weakly compact on VMOA and because VMOA** = BMOA and $\mathsf{J}_g^{**} = \mathsf{J}_g$ it follows that $\mathsf{J}_g(\mathrm{BMOA}) \subset \mathrm{VMOA}$. In particular $\mathsf{J}_g(f_0) \in$VMOA so that $|\mathsf{J}_g(f_0)'(z)|^2(1-|z|^2)\,dm(z)$ is a vanishing Carleson measure. The vanishing of the second term above as $n\to\infty$ follows and the proof is complete.

Conversely suppose $g \in LMOA_0$. Then there is a sequence of polynomials $\{p_n\}$ such that $\lim_{n\to\infty}\|g-p_n\|_{LMOA} = 0$. Because the integration operator is compact in $BMOA$ we have from Proposition (2.2) that J_{p_n} is compact for each n. Also

$$\|\mathsf{J}_g - \mathsf{J}_{p_n}\| = \|\mathsf{J}_{g-p_n}\| \le C\|g-p_n\|_{LMOA} \to 0,$$

so that J_g can be approximated by compact operators hence it is compact. The proof is complete. □

It is clear from the above and from $\mathsf{J}_g^{**}|_{VMOA} = \mathsf{J}_g$, that J_g is compact on VMOA if and only if it is compact on BMOA if and only if (3.6) holds.

Question. Are there functions g such that J_g : VMOA $\to$ VMOA is weakly compact but not compact?

4. Remarks

The spaces of symbols g which satisfy (3.3) or (3.6) are weighted BMO spaces and have appeared in the literature in connection with Hankel operators [JPS] and with multipliers of BMOA [Ste]. Let $g \in H^2$ and $P : L^2(\partial\mathbb{D}) \to H^2$ be the natural orthogonal projection. The Hankel operator H_g defined initially on bounded analytic functions f is $H_g(f) = P(g\bar{f})$. The theorem of Nehari says that H_g can be extended to a bounded operator mapping H^2 into itself if and only if $g \in$BMOA. The same result holds if H^2 is replaced by H^p, $1<p<\infty$, but at the endpoints H^1 and BMOA the characterization is different. It was shown in [JPS] that H_g is bounded on H^1 or on BMOA if and only if

$$\sup_{I\subset\partial\mathbb{D}}\left\{\frac{\log\frac{2}{|I|}}{|I|}\int_I |g(e^{i\theta}) - g_I|\,\frac{d\theta}{2\pi}\right\} < \infty, \tag{4.1}$$

where $g_I = \frac{1}{|I|}\int_I g(e^{i\theta})\frac{d\theta}{2\pi}$. Cima and Stegenga in [CiSte] obtained a variant of this characterization of bounded Hankel operators on H^1 by a condition which is identical with (3.3). We used their techniques in the proof of Theorem (3.1) above.

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