Deepening Teacher Understanding of Fundamental Concepts in Mathematics

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Deepening Teacher Understanding of Fundamental Concepts in Mathematics

by
Meghan Carr
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A thesis submitted to the
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Deeping Teacher Understanding of Fundamental Concepts in Mathematics

by

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[Signatures and dates]
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Chapter 1

Introduction
Problem Statement and Significance

Recent research has shown that conceptual mathematical content knowledge of teachers in the United States is significantly deficient.

The thoroughness of teacher content knowledge of mathematics is one of the most significant problems in today's educational community. While student performance and understanding is a daily concern for teachers, researchers, parents, politicians, and businesses, how can one look to improve student learning without considering the major source of their learning: their teacher? Research has logically concluded that in order for teachers to promote solid conceptual understanding in their classrooms they themselves must have a deep conceptual understanding of the content. In addition, research has shown that the depth of a teacher's knowledge in mathematics directly affects their expectations for students, as well as their instructional methods. It is without question that teacher content knowledge is a foundation for which we build much of our educational belief system on. Teaching-for-understanding, questioning, creating connections, promoting critical thinking, encouraging reasoning and proof, and requiring mathematical accuracy are only some of the many beliefs and methods that are popular among educators. While these techniques are intended to guide students to a truly conceptual understanding of mathematics, a prerequisite to realizing these theories of teaching is thorough teacher content knowledge.

Until recently, it has been assumed that all teachers have the conceptual understanding necessary in order to teach effectively. Unfortunately, this is not the
case. In order to improve student performance and understanding it is necessary to recognize, address, and improve the mathematical content knowledge of teachers.

**Purpose and Rationale**

The purpose of this thesis is to begin the process of recognizing, addressing, and improving the mathematical content knowledge of teachers. The research will observe how professional development that focuses on fundamental mathematical concepts deepens the conceptual understanding of teachers, as well as improves their instructional methods. The research will also serve to enlighten teachers to the importance of actively studying the math content that they teach.

The motivation behind this research is *not* to investigate *if* deficiency in teacher content knowledge is a problem, but to recognize that it *is* a problem, and to address this problem. Therefore, the first and foremost goal of the research is to guide teachers to a deeper understanding of fundamental mathematical content. Ultimately, however, the goal of the research is to set in motion for teachers an ongoing dedication to the study of the mathematical content that one is expected to teach, and to let this deep understanding be the driving force when developing instructional techniques.

**Definition of Terms**

One of the major goals of this research is to promote a deep understanding of fundamental mathematics. Throughout this research "deep understanding of fundamental mathematics" will be defined as Liping Ma defines Profound Understanding of Fundamental Mathematics (PUFM). In order to develop a deep understanding of fundamental mathematics, one must demonstrate connectedness
among various mathematical concepts, multiple perspectives that recognize advantages and disadvantages to various approaches to a problem, an ability to distinguish the "basic" skills in a problem, and a longitudinal coherence that allows for comparisons to previous and subsequent concepts (Ma, 1999, p. 122).

Fundamental mathematics is defined as the arithmetic and geometry that is learned prior to pre-algebra. In the United States, this occurs in grades 1 through 6. Therefore, it is elementary school teachers who are often responsible for the teaching of fundamental mathematical concepts.

Another goal of this research is to "improve instructional methods". This essentially means aligning instructional methods with the content so that the content is driving the instruction. Instructional techniques that involve practicing or learning arbitrary procedures will be modified so that the instruction emphasizes the rationale behind the algorithm. Instructional methods that teach procedures will not be eliminated, but instead will shift so as to focus on the "why" rather than simply the "how". "Improved instructional methods" is fine-tuning the instructional method so that they actually accomplish the goal of teaching conceptual understanding. The improvement of instructional methods also includes diversifying the methods a teacher uses to explain a certain concept.

**Summary**

Of the many problems and concerns that are currently facing educators in America, the insufficient mathematical content knowledge of teachers is perhaps the most disconcerting, as teachers are the foundation on which most of education is built. It is encouraging, however, to know that by addressing this problem,
significant improvements can be made in mathematics education. A vast amount of current research explores the dimensions of this problem, as well as implications for reform.
Chapter 2

Review of Literature
Introduction

Today’s world of math education is in the midst of addressing a variety of significant concerns. Through recent years much controversy has centered on the “Math Wars” and what constitutes the “best instructional practices” in the field of math education. Despite the importance of researching the pedagogical methods that help students learn most effectively, there is a growing feeling among researchers and math educators that a fundamental piece of the puzzle has been left out: the content. This review of literature will paint a picture of the current research describing teachers’ math subject knowledge, particularly for elementary teachers, and its effects on the learning of mathematics.

“He who can, does. He who cannot, teaches.”

This comment originally made by George Bernard Shaw, and highlighted by Shulman (1986), is an insult to all those involved in the profession of teaching (p. 4). However, researchers and educators alike are beginning to recognize the need for additional mathematical content support for teachers of all grade levels. Studies have shown that American teachers and students perform significantly lower on mathematics achievement tests than their international peers. Although the concern focuses on elementary teachers’ mathematical knowledge, this topic is relevant to all math teachers, as they continually build upon their students’ prior knowledge. Recent studies have analyzed many aspects of this concern including background and historical data on the importance of teacher content knowledge, theoretical explanations that strive to define “subject matter knowledge”, and the recognition of assumptions that perpetuate the cycle of lack of content knowledge. Recent studies
have investigated not only the achievement levels of American students and teachers, but have also identified the impact of teacher beliefs, the impact of lack of content knowledge on instructional practices, the specific weaknesses in content, and the lower expectations held by students and teachers in America. Most researchers also offer suggestions for reform including teacher-education changes, an increase in reflective time offered to educators, and the hiring of a math specialist. Conclusions and implications will be discussed throughout this chapter which will lead to a description of a focused research hypothesis.

**Background**

The background information that can be found on teachers' content knowledge is relatively scarce, however interesting and significant. As with many issues in education, the importance of teachers' content knowledge has been addressed throughout history and has acted as a pendulum, swinging from one theoretical extreme to another. Available literature that describes the historical perspective of this issue is helpful in understanding the current situation. Also helpful in unraveling the complexities of the issue of teachers' content knowledge are literatures that acknowledge and clarify the ambiguity concerning what constitutes subject matter knowledge. Finally, researchers are beginning to analyze the issue by first recognizing and contesting some widespread assumptions about content knowledge in elementary mathematics.

**History**

The idea of using tests to assess teaching skills is not a new concept in America. What is relatively new is the concentration of these tests on the theory and practice of
teaching. For instance, the California State Board Examination for Elementary School Teachers given in 1875 consisted of 1000 points, only 50 of which focused on the knowledge of teaching (Shulman, 1986, p. 5). The remaining 950 points focused on content, such as written arithmetic, mental arithmetic, and algebra. Similarly, Shulman's research indicates that all studied tests from this time period, including state assessments from Massachusetts, Michigan, Nebraska, and Colorado, consisted 90 to 95% of content-oriented questions. The tests focused on the knowledge base that was assumed to be needed by teachers. Today's standards, on the other hand, emphasize the assessment of the capacity to teach, such as basic abilities to read, write, spell, and calculate, rather than demonstrating the knowledge of the curriculum content (Shulman, 1986, p. 6). For instance, the following categories for teacher review and evaluation were recently proposed in planning for a state-wide teacher evaluation: Organizing, recognizing individual differences, cultural awareness, understanding youth, management, and educational policies and procedures (Shulman, 1986, p. 5). While it is important for teachers to be aware and proficient in these areas, it is alarming and concerning that the focus of teacher assessment could shift so far away from content. Similarly, Marc Swadener's (1978) research showed that in 1975, twenty-seven of the states did not specifically mention any mathematics as a requirement for elementary teacher certification (p. 676). What happened to the subject matter?

The change in state requirements for teacher certification is only one of the indicators of a shifted focus in education. Also worth noting is the transformation of the National Council of the Teachers of Mathematics (NCTM). In the 1920's, NCTM
held ties to the Mathematical Association of America, and was led by content-oriented math teachers (Siegel, 2006, p. 4). However NCTM is now an organization led by professors of math-education and has developed a new vision of mathematics learning and curriculum. Instead of focusing on what to teach, the council addressed how to teach and how to assess student progress (Siegel, 2006, p. 3).

Although current literature strives to be unbiased in its opinion on the debate between "traditional" math and "reformed" math, the dispute is a classic example of the swinging pendulum in education: In less than a century, state requirements and NCTM developments have changed drastically in their view of the importance of math content knowledge of teachers. Shulman (1986) calls this lack of attention to teachers' content knowledge the "missing paradigm" (p.6). Research has been thorough in analyzing and testing effective pedagogical strategies, however it is lacking in the analysis of content-oriented questions such as:

"Where do teacher explanations come from?  
How do teachers decide how to represent what they teach?  
How do they decide how to question students about it or how to deal with problems or misunderstandings?  
What are the sources of analogies, metaphors, examples, demonstrations, and rephrasing?" (Shulman, 1986, p. 8)

The consideration of these questions is in fact prerequisite to acquiring successful instructional methodologies. A teacher cannot successfully demonstrate the widely appraised instructional techniques such as "check-for-understanding" or "questioning to develop critical thinking" unless they have a sound, deep understanding of the concepts being studied. Through Liping Ma's (1999) study of the content knowledge of elementary teachers, she concluded that "even a strong belief of 'teaching mathematics for understanding' cannot remedy or supplement a teacher's
disadvantage in subject matter knowledge" (p. 36). Although the teachers had intentions of teaching for conceptual understanding, their belief of teaching could not be realized. This “missing paradigm” needs to be addressed in current research. It is clear to see that it is time for a reanalysis of the balance of pedagogical skills and content knowledge.

Assumptions

Another relevant piece of background knowledge needed in order to understand the complexities of this issue is the recognition of the assumptions that are made about learning to teach math. Deborah Ball highlights three of these assumptions.

First of all, it is assumed by many people, including educators, that traditional school math is not difficult (Ball, 1990, para. 462). This is a common, but dangerous assumption. Although the procedures may be basic, the understandings of the fundamental math concepts taught in early grades are complex and challenging. As Liping Ma (1999) states, "in the United States it is widely accepted that elementary mathematics is "basic", superficial, and commonly understood (p.146). As Ma's data reveals, however, anyone who teaches elementary mathematics has to study it hard in order to understand it in a comprehensive way.

The second assumption that Ball (1990) recognizes is that pre-college education provides teachers with much of what they need to know about math. This is based on the mistaken assumption that “if you can do it, you can teach it” (p. 462). The third assumption, which is related to the second, is that majoring in mathematics ensures subject matter knowledge (Ball, 1990, para. 463). Both the second and third
assumptions are based on an avoidance of the problem; at every step along the way, it is assumed that teachers learn the content knowledge (the knowledge that they are expected to teach) somewhere else. In college it is assumed that prospective teachers have learned the content in elementary and secondary school, and while teaching it is assumed that they have acquired the content knowledge in college.

When looking at the big picture, there are few places where elementary teachers are actually taught the understandings that they are expected to know. These assumptions can be summarized into an overall need for prospective teachers and colleges to first realize the depth of what they must know about math, and then to actively address the problem.

Theory: What is “subject knowledge”?

A third relevant discussion in understanding the background of the issue of teachers’ content knowledge is the theory behind what constitutes having subject knowledge. Deborah Ball (1990) acknowledges that there is little agreement about what is meant by “subject matter knowledge for teaching” (p. 450). It is easiest for researchers to measure subject matter knowledge using concrete data such as grade point average, test scores, major fields of study, and courses taken. For instance, in Dora Skypek’s (1965) research she defined “Measures of Mathematical Competence” as the number of years of high school mathematics completed, grade averages in high school, SAT-Math scores, and scores on the college entrance examination in math (p. 771).

On the other hand, some researchers describe “having content knowledge” using a less quantitative definition. Carol Aubrey (1996) defines it as the knowledge
of math, knowledge of teaching, and knowledge of children's cognitions, while Shulman (1986) defines it as the amount and organization of knowledge (p. 183, p. 9). Liping Ma (1999) has defined "having content knowledge" as having a Profound Understanding of Fundamental Mathematics (PUFM), which embodies "an understanding of the terrain of fundamental mathematics that is broad, deep, and thorough" (p. 124). Ma (1999) explains that the four components of PUFM are connectedness, multiple perspectives, basic ideas, and longitudinal coherence (p. 122). Teachers at each grade level need to understand what has gone before and what will come after the mathematics they are teaching. Deborah Ball (1990) also gives a similar definition of a teacher who has a "substantive knowledge of mathematics" as one who is correct in their knowledge of concepts and procedures, understands the underlying principles and meanings, and appreciates and understands the connections among mathematical ideas (p. 458). Teachers should understand the subject in depth in order to represent it appropriately and in multiple ways with story problems, pictures, and concrete materials. Taken as a whole, it appears that the majority of research literature on the topic of teachers' mathematical content knowledge defines "subject matter knowledge" as more than a fluency in mathematical procedures. The literature is emphasizing the need for a deeper understanding that would enable teachers to make connections, expose students to various representations, and develop probing questions that challenge students to see the deeper meaning of math concepts.

Data

Recent studies that have been conducted on the issue of teachers' mathematical subject knowledge have consisted of both quantitative and qualitative
data. Most of the research involves the comparison of American students and teachers to foreign students and teachers. The foreign countries appearing most frequently in the studies are those from East Asia, as these countries have achieved at the highest levels on international mathematics competency tests. Quantitative studies have relied mainly on test scores to assess student and teacher competency whereas qualitative studies often use interviews, observations, and questionnaires.

Quantitative Studies

Two quantitative studies that include an analysis of the mathematics content knowledge of teachers are the Stevenson et al. (1990) study that compared students in Chicago and Beijing on their mathematical ability, and the Third International Mathematics and Science Study (TIMSS) analyzed by Hiebert, et al. (2005) that compared the math competency of 8th-graders from 7 countries around the world. Stevenson's et al. (1990) results showed that the children in Chicago performed consistently lower than their peers in Beijing (para. 1053). The TIMSS study showed that the average score for United States students on an eighth-grade assessment was lowest among 7 countries (Hiebert, 2005, p. 117). One of the reasons for the lower achievement is thought to be the lack of rigor, partially due to the lack of content knowledge of teachers (Hiebert, 2005, para. 116.) Quantitative results have shown that American teachers and students fall behind their international peers in math competency.

Qualitative Studies

While the quantitative data found by research on this topic is informative, it is qualitative data that will paint a more descriptive picture of the nature of the problem.
Foss and Kleinsasser (1996), Gellert (1998), Aubrey (1996), Ball (1990), and Ma (1999) have each conducted studies that analyze both the perspectives that elementary teachers have of math education as well as their comprehension of mathematical content.

Foss and Kleinsasser's (1996) study revealed a strong relationship between a teachers' views of content knowledge and their instructional actions, which was concerning because the teachers' views of math content tended to be vague and unclear (para.440). When asked about their view of math, some teacher responses were:

"The process of coming to an answer through computation or, I guess, through computation and different ways of finding the answer.

Numbers, its just numbers and the outcomes of numbers put together or subtracted or whatever. There's lots of different ways." (Foss & Kleinsasser, 1996, p. 434)

As Foss and Kleinsasser (1996) conclude, the teachers could not initiate learning of mathematics when they believed math was merely a collection of methods (p.441).

Uwe Gellert's (1998) study found similar results. Through journal entries of 42 prospective elementary teachers, Gellert (1998) found the views of mathematics education to be largely superficial, with emphasis placed making their math classroom "fun" as well as a lack of conscious analysis of mathematical concepts (p. 37). When asked to describe their view on the teaching of mathematics, some responses were as follows:

"In principle, I want to try to teach the children mathematics in a playful way."
In mathematics classes, mathematics should be wrapped in a way that students do not become aware of the fact that mathematics is being taught.

There is a radio play cassette...which imparts arithmetic to the children by means of cooking...One need not tell them that they actually calculate.” (Gellert, 1998, p. 33)

Gellert (1998) found that teachers are focused on student entertainment rather than mathematical foundation, with a view that mathematics is something terrible that one does better to disguise (p. 37). What these teachers are lacking is a value of “mathematical archaeology”, “the basis for discussing the math found and for consciously transferring it” (Gellert, 1998, p. 37). The absence of this threatens to trivialize the meaning of math education. In general, these participants failed to show a conceptualization of mathematical content.

Slightly different from the above-mentioned studies, Aubrey’s (1996) study connects the lack of content knowledge of teachers to its effects on their instructional practices. Through both interviews and classroom observations of four elementary school teachers, she found that the teacher’s subject content knowledge had a large impact on their practice (Aubrey, 1996, para. 181). For instance, Teacher D had a strong foundation of content knowledge and was therefore confident in setting up explorations for students in many lessons. This teacher could also represent the concepts in pictures, diagrams, and models (Aubrey, 1996, p. 192). Teacher C, on the other hand, lacked a firm grasp of the subject matter and was therefore unable to develop explanations or questions that would lead students to discover meaningful mathematics. Teacher C showed less interaction with content throughout their lessons and instead placed more emphasis on basic ideas of numbers rather than on
classification, shape and space, measurement of quantity, and data handling (Aubrey, 1996, p. 192). It is clear that the depth of content knowledge directly affected the instructional choices of the teacher.

It is concerning to see that a teacher’s content knowledge will be influenced by their beliefs about the subject, and furthermore their instructional capabilities are reliant upon their content knowledge. With the beliefs of teachers, rather than the mastery of content, driving a teacher’s instructional methods, the practice among the different teachers varies in terms of the math content the teachers chooses to introduce, its representation in tasks, and consequently, in the quality of instruction provided (Aubrey, 1996, para.194).

Similarly, Ma's (1999) study indicated a strong correlation between teacher knowledge and teacher's expectation of students as well as instructional technique (para. 52). Teachers with a strictly procedural understanding of the content diagnosed student misconceptions as procedural mistakes, and used instructional techniques that emphasized the (many times seemingly arbitrary) procedure (Ma, 1999, p. 54). For instance, one teacher who did not have a conceptual understanding of multi-digit multiplication diagnosed a student error as a confusion of where to "line up" the numbers (Ma, 1999, p. 34). Although this teacher used the terminology "place value", she used this not as a mathematical concept, but for labeling each column. This teacher also offered an instructional technique of using something that "would catch the student's eye", such as drawing elephants for placeholders (Ma, 1999, p. 35). Although "interesting", this teaching strategy implies that the procedure is arbitrary, and it fails to promote any meaningful mathematical learning. This
reemphasizes the above-mentioned results in that a teacher's views and knowledge directly affects their teaching strategies as well as student learning.

The studies of both Deborah Ball (1990) and Liping Ma (1999) also focus on specific content including division by fractions, a fundamental concept that students often struggle with. Ball (1990) analyzed the results from questionnaires and interviews with 252 prospective teachers. The questionnaire item consisted of a multiple-choice problem that asked elementary teachers to circle all story problems that represented $\frac{1}{4} + \frac{1}{2}$. Only 30% of the elementary teachers circled the correct response, and of those that circled correct, many circled an incorrect answer as well (Ball, 1990, p. 454). The interview task asked teachers to perform $1\frac{3}{4} + \frac{1}{2}$ and to then generate a story problem representation of this. No elementary teachers could generate a story problem (Ball, 1990, p. 454). These teachers had much difficulty “unpacking” the meaning of division with fractions.

Likewise, Ma (1999) analyzed results from interviews with 23 American elementary teachers and 72 Chinese elementary teachers. One of the interview questions asked the teachers to calculate $1\frac{3}{4} + \frac{1}{2}$ and to then create a story problem that would represent this calculation. Only 43% of the American teachers successfully calculated the answer and only one provided a conceptually correct but pedagogically problematic representation (Ma, 1999, p.56). All 72 Chinese teachers, however, calculated the problem correctly and 90% were able to provide a suitable representation (Ma, 1999, p.64).

According to recent studies, American elementary teachers’ math understandings tended to be rule-bound and thin. None of the American teachers
earned Ma’s (1999) distinction of Profound Understanding of Fundamental Mathematics (p. 129). Ball (1996) concluded that the mathematical understandings that prospective teachers bring are inadequate for teaching mathematics for understanding (para. 464).

**Differences in Expectations**

Many researchers admit that when comparing America to other countries in terms of academic achievement, the U.S. faces challenges due to the vastly different expectations of the American society. Amidst lower test scores, satisfaction with schools is higher among American parents while lower among Japanese and Tiawanese parents (White, 1993, p. 534). Another study shows that American students suggest they like math, believe they are doing well in math, and do not perceive math as a difficult subject, while test scores show that they are underachieving (Stevenson et al., 1990, p. 1062). The relatively unjustifiable positive self-evaluation of American students reflects the lower standards held for children’s performance in America than in China. Furthermore, mathematics held a lower status in the eyes of the American teacher than of the Chinese teacher. When asked if math was the most important subject to teach, 9% of American teachers said 'yes', while 34% of Chinese teachers said 'yes' (Stevenson et al., 1990, p. 1065). In order to raise the expectations and success levels among students, the attitudes and achievement levels among teachers must first be addressed.

**Suggestions for Reform**

In light of the concerns discussed thus far, most literature on the subject of teachers’ content knowledge offers suggestions for reform.
The first major suggestion mentioned in almost all of the literature is a need to refocus teacher-education. As Ma (1999) states, teacher education is a way to break the "vicious circle formed by low-quality mathematics education and low-quality teacher knowledge of school mathematics" (p. 149). Prospective elementary teachers are taught methods for teaching, but they rarely study content. This is partially due to the assumption that this content is learned and understood elsewhere. Ma (1999) emphasizes that it makes sense to address this concern on a collegial level, as most teachers in the United States attend college (para. 149). A similar yet more specific recommendation that some researchers are offering is increased minimum requirements in mathematical competence for graduation from teacher education (Skypek, 1965, 772). In general, the feel is that subject matter preparation for prospective elementary teachers needs to be the central focus in teacher education.

The second major suggestion in improving elementary teachers’ math competence is to allow teachers more time for class preparation. One of the teachers who earned Ma’s (1999) Profound Understanding of Fundamental Mathematics emphasizes the importance of having time to reflect on the content:

“I always spend more time on preparing a class than on teaching, sometimes three even four times the latter. In a word, one thing is to study whom you are teaching, the other thing is to study the knowledge you are teaching. If you can interweave the two things together nicely, you will succeed. We think about these two things over and over in studying teaching materials. Believe me, it seems to be simple when I talk about it, but when you really do it, it is very complicated, subtle, and takes a lot of time. It is easy to be an elementary school teacher, but it is difficult to be a good elementary school teacher.” (Ma, 1999, p.135)

Ma (1999) also states that having a deep conceptual understanding of a topic is a "result of deliberate study" (p.22). This is good news for teachers who have yet to
attain a PUFM! A deep, conceptual understanding of math does not happen magically, but instead there is a course of action that can be taken to reach it.

A third, lesser mentioned but valid suggestion is that elementary schools should employ math specialists to either do the teaching of the math, or to guide the teachers in their understanding and teaching of the math. Chinese mathematics teachers are specialists and therefore have more time and motivation for developing their understanding of mathematics (Howe, 1999, p. 584). Practically speaking, this suggestion seems reasonable in that it would require only the more qualified people to teach math, and it would also raise the incentives for mathematically inclined people to become teachers.

**Conclusion**

Through the historical evidence it is clear that American education has swung to a pedagogical extreme and, due to common mistaken assumptions about the rigor of elementary mathematics, the lack of mathematics content knowledge of teachers is only just beginning to draw public attention and concern. Through studies we have seen that the beliefs and knowledge of elementary teachers are tending to drive their instruction and that their understandings of fundamental math concepts are in need of guidance and depth. Although the suggestions of researchers are possible and encouraging, America faces the obstacle of lower educational expectations. Nevertheless, this issue is recently becoming widely recognized and attempts at reform are possible and necessary.

In the midst of the popular debate on “traditional” math versus “reformed” math, the following comment by Alan Siegel (2006) stands out: “A deep
understanding of what is being taught should be used as a prerequisite for deciding how to teach a particular topic” (p. 8). This comment alludes to the idea that an understanding of the content is prerequisite to choosing and implementing pedagogical strategies.

**Research Hypothesis**

The action research that follows this literary review is based in the belief that a deep, comprehensive knowledge of the content is actually one of the most valuable pedagogical techniques; it is the foundation for questioning, application, and engaging. The hypothesis is that through professional development that explores fundamental mathematics, and through reading Liping Ma's (1999) book, *Knowing and Teaching Elementary Mathematics*, teachers will deepen their understanding of fundamental mathematics, learn improved and diverse instructional methods, as well as appreciate the importance of studying thoroughly the mathematics that they teach.
Chapter 3

Development of Hypotheses and Outcome Measures
The first hypothesis of this study is that teachers will attain a deeper understanding of fundamental mathematics through participating in the professional development workshop entitled "Deepening Fundamental Mathematics" and reading Liping Ma's book entitled Knowing and Teaching Elementary Mathematics.

Outcomes will be measured through a pre- and post-assessment in which teachers will demonstrate their understanding of four fundamental mathematics concepts. After implementing the workshop, the following effects will be noted: Do teachers perform at a higher level on the given assessment of four mathematical concepts? This hypothesis will also be measured using a workshop evaluation survey in which participants will rate how well they felt the workshop met its goal of deepening their understanding of mathematics.

The second hypothesis of this study is that teachers will improve their instructional methods through participating in the workshop and reading Ma's book.

The third hypothesis is that teachers will have begun to develop a dedication to the deliberate study of the mathematics they teach by reading the book and participating in the workshop.

Both the second and third hypotheses will be measured by the evaluation survey given at the end of the last workshop session. This survey asks teachers to rate numerically the effectiveness of the workshop at deepening their mathematical understanding, improving their instructional techniques and at setting into motion a dedication to the deliberate study of the mathematics they teach. The survey also asks teachers to comment on their rating. After implementing the workshop, the following effects will be noted: Do teachers rate the workshop as 3 or 4 (having "met
its goal” or having “met its goal very well”) for the three goals? Also, do the teachers’ comments describe positive effects of the workshop and/or readings on their improved content knowledge, instructional methods, and dedication to studying the math they teach?

Although formal observations of participants will not be used as outcome measures, informal observations of participants will be used to modify activities and lessons in upcoming workshops to better meet the needs of the group.
Chapter 4

Methods and Procedures
The procedure for intervention of this action research focuses primarily on a professional development-style workshop that the researcher designed and implemented. Although the significance and rationale behind this action research has been established, the creation and execution of the workshop sessions involves detailed steps and decisions that require explanation.

Offering the Workshop

This workshop was offered via an email flyer sent to teachers in the Penfield Central School District who teach or support math instruction at any grade level. Teachers were offered up to 6.5 hours of paid professional development credit. The workshop was located in a classroom at Penfield High School, at a time late enough in the day so that teachers from any building in the Penfield School District could attend. Teachers were notified in the flyer that they would read and discuss Liping Ma’s book Knowing and Teaching Elementary Mathematics, and that the workshop would entail 5 sessions (each an hour and 15 minutes) that explored four fundamental math concepts (subtraction, multiplication, division by fractions, and the connection between area and perimeter).

Participants

Sixteen teachers signed up for the workshop, although only nine of these teachers participated in all five sessions. This action research will analyze only the data of the nine participants who completed the workshop in its entirety. Of the nine teachers who participated, one is a high-school teacher, four are middle-school teachers, and four are elementary-school teachers. The high school teacher has been teaching for 5 years and has taught ninth, tenth, eleventh, and twelfth grade math.
Two of the middle school teachers have been teaching for over 20 years, while the other two middle school teachers have been teaching for less than 6 years. Of the four elementary school teachers, one teaches first grade, one teaches fourth grade, and one taught fifth grade in prior years and now is the department chair. Both the first and fourth grade teachers have been teaching for less than 6 years. The last elementary school teacher is currently a reading specialist, but performed as a special education math support teacher for many years.

**Development of the Workshop**

The workshop was titled “Deepening Fundamental Mathematics – A Book Study”. This was to convey the fact that the major goal of the workshop was to deepen mathematical understandings of teachers and that the workshop was intended to align with Liping Ma’s book *Knowing and Teaching Elementary Mathematics*. The four mathematical concepts that Ma’s book focuses on, that the workshop focused on as well, are as follows: subtraction with regrouping, multi-digit multiplication, fractional division, and the relationship between area and perimeter, which also involves elementary ideas of proof. Due to the needs of the group, workshop session 3 included discussions and activities surrounding the concept of zero which is not included in Ma’s book. A summary of each workshop session (stage of intervention) is described below. (See Facilitating Guides in Appendix D, p. 89, 94, 101, 105, 116)

Session one began with stating of the goals of the workshop. It then provided a presentation that described the significance and background of the research question, and then an introduction to Ma’s study. The participants were then given 25
minutes to complete the pre-assessment. (See Appendix A, p. 70) The pre-assessment and post-assessment, which consist of the same four questions, were constructed in alignment with the four questions that Ma used in her study. Each of the four questions tests a mathematical concept that is taught in the elementary grades, and each is considered a fundamental mathematical concept. The first concept tested is subtraction with regrouping, the second concept tests multi-digit multiplication, the third concept tests fractional division, including generating representations, and the fourth concept tests exploring new knowledge through the relationship between perimeter and area. All four questions are asked in a pedagogical context. After the pre-assessment, teachers participated in an activity in small groups where they were asked to solve the subtraction-with-regrouping problem $\frac{63}{-29}$ in as many ways as possible. The teachers were then asked to consider the problem $\frac{15}{-7}$. The purpose for this second problem was to force teachers to elaborate upon methods used in the first problem and come up with alternatives to "borrowing", or "counting on fingers". To wrap up session one, participants were assigned to read Chapter one and given "Guiding Questions" to help extract from the reading the goals of the workshop. (See Guiding Questions for Chapter #1 in Appendix D, p. 93)

After a brief discussion on overall impressions of the book, session two required teachers to record their thoughts on what "deep understandings of mathematics" are essential to students when learning subtraction with regrouping. (See Recording Sheet in Appendix D, p. 97) After this was discussed, the
participants were presented with a summary of the different methods of subtraction with regrouping that they had come up with during the end of the last session. In small groups teachers discussed whether any methods were “missing” from the list. They also discussed how the methods did or did not promote the same understandings, and whether each method would support students in developing all of the deep understandings discussed earlier. Teachers concluded their small group discussions with a conversation on preferred instructional methods for teaching subtraction with regrouping. (See Methods of Subtraction with Regrouping in Appendix D, p. 98) As a large group, participants were then asked to analyze the three major components of Ma’s proposed “knowledge package” of subtraction with regrouping: Subtraction with minuends between 10 and 20, subtraction with minuends between 19 and 100, and subtraction with minuends larger than 99. Participants were assigned to read chapter 2 and given the guiding questions as they left this session. (See Guiding Questions for Chapter #2 in Appendix D, p. 100)

Session three was initially intended for discussing the concepts involved in multi-digit multiplication, although because most participants demonstrated depth of knowledge of this concept on the pre-assessment, this discussion was shortened and extra time was spent delving into the concept of the meaning of zero. This decision was also made in order to accommodate the high school and middle school teachers’ desires to explore higher level concepts. After a discussion on the instructional methods used in the book to correct the student misconception, and a discussion on the difference between teaching conceptually versus procedurally, a debate ensued when teachers were posed the question “How important is it that zero be used as a
placeholder?" While some teachers insisted that 0 is the necessary symbol, other teachers argued that 0 only represented a placeholder in this case and it was not necessary that it be written down as long as place value was maintained. This led to more discussion on the meaning of zero, and teachers were asked to brainstorm the meaning(s) of zero in their small groups. Session 3 concluded with the analysis of a student misconception: A student concludes that 5.20 is larger than 5.2 because 20 is larger than 2. Teachers were asked to answer the following question: "What is the student’s misconception and how would you go about correcting it?" As teachers left session 3, they were assigned to read chapter 3 for next session and were given the guiding questions. (See Guiding Questions for Chapter #3 in Appendix D, p. 104)

Session four was very important for many participants because it included the analysis and discussion of fractional division. Many participants struggled with this topic on the pre-assessment, and therefore this session began by analyzing the algorithm “Multiply by the reciprocal”. (See Making Sense of the Algorithm Sheet in Appendix D, p. 109) Teachers were then presented with a review of the models of division by fractions, and then asked to read and discuss case analyses in which students have tried to represent a particular fractional division problem as a word problem. (See Different Models of Division by Fractions Sheet in Appendix D, p. 112) Teachers had to determine if in fact the students’ representations were correct, and if so, what model of division by fractions was used. Teachers were then asked to create word problems of their own to represent the fractional division problem. Participants discussed how fractional division relates to concepts that students have learned previously. Lastly, teachers watched a video clip of a student demonstrating
fractional division and answered the question “What is the nature of the student’s understanding of fractional division?” (See Video Clip Sheet in Appendix D, p. 114) Teachers were handed the guiding questions for chapter 4 as they left the session. (See Guiding Questions for Chapter #4 in Appendix D, p. 115)

The fifth and last session began with a discussion on the importance of confidence in math, and teachers were then asked to discuss in small groups the importance of “shaping” a student’s thinking. This discussion led to a large group discussion on the habits of mind, including those used in the exploration of the relationship between perimeter and area. The teachers were then presented with a summary of chapters 5, 6, and 7 from Ma’s book, which discuss the Profound Understanding of Fundamental Mathematics (PUFM), when PUFM is attained, circumstantial differences between teaching in the United States versus teaching in China, and the value of the text book and curriculum in gaining PUFM. Lastly, teachers were given 25 minutes to complete the post-assessment and workshop evaluation survey. (See Appendix A and B, p.70, 72)

**Design**

This research uses both quantitative as well as qualitative data. Quantitative data was gathered from the participants’ scores on the pre- and post-assessment, which was graded using multiple rubrics. Quantitative data was also gathered from the rating scales on the workshop evaluation survey. The qualitative data was gathered from the work shown on the pre- and post-assessments, participant comments on the survey, and informal observations of participants and their discussions.
Both methods of data collection were used to accommodate the unique nature of this research design, which is both flexible and rigid in nature. It is flexible in the sense that the creation of activities and prompting discussion questions for each workshop session were accommodating to the needs of the audience. This flexibility, however, did not jeopardize the validity of the research. The structured parts of the research design included set goals surrounding a focus on conceptual understanding of mathematics topics with exploration of instructional methods. It was necessary that all workshop sessions be created and facilitated according to these goals. The design of this research is also rigid in its expectation that the workshop partner with Ma's book, specifically chapters 1 through 4. Chapter readings were assigned weekly, regardless of participants' personal opinions of the book. The design of this research was meant to be responsive to the needs of the participants while at the same time providing the structure to derive valid conclusions.

Validity

It is important to understand that the design of the research promotes internal validity of its results. First of all, because participants are given a pre-assessment prior to reading the book and participating in workshop activities, the research design is able to account for current knowledge and understanding and will not misinterpret this initial understanding as effects of the workshop or readings. Secondly, because the workshop was offered to teachers of all grade levels, and the research does in fact include teachers of a variety of grade levels, the audience is well distributed.

This research does face challenges to its validity, however. Although the selection process included an invitation to teachers of all levels, only teachers who
chose to participate were included in this research. This could bias results in that teachers who are likely to sign up for this workshop may not be a random sampling. Teachers who are stronger at mathematics may tend to be interested in the workshop because it involves a study of mathematics. On the other hand, teachers who are weak in mathematics may respond to the part on the flyer which describes that the workshop is intended to help strengthen teachers' mathematical content knowledge. Also, because this research design requires that teachers do outside learning in the form of reading the assigned chapters, the researcher has no way of knowing if participants are completing this assignment. The design does address this issue, however, by providing guiding questions with the assignment of each reading.
Chapter 5

Data Collection
Pre- and Post-Assessment

After the workshop concluded, both pre- and post-assessments were graded based on a rubric specific to each problem. Because it is hypothesized for each problem that teachers will exhibit a deeper understanding of content on the post assessment due to the effects of the workshop and reading, the major purpose of the rubrics was to evaluate the depth of understanding of each concept. Problems #1 and #2 were based on the four components of attaining a Profound Understanding of Fundamental Mathematics (PUFM) as defined by Liping Ma: distinguishing the “basic” skill, demonstrating connectedness, recognizing multiple perspectives, and using longitudinal coherence. Problems #3 and #4 were graded on customized rubrics due to teacher responses and question format.

Problem #1

How would you explain to a group of second graders how to solve the following problem?  

\[
\begin{array}{c}
52 \\
-27 \\
\end{array}
\]

As stated above, the rubric for problem #1 contained the components of PUFM, with one point awarded for each element displayed: One point for distinguishing the “basic” skill of regrouping (a purely procedural description was insufficient), one point for demonstrating connectedness between subtraction with regrouping and another mathematical concept, one point for recognizing multiple perspectives of subtraction with regrouping, and one point for using longitudinal coherence in their explanation, which links subtraction with regrouping to prior
and/or subsequent topics. Teachers could earn a maximum of four points on this problem, and a minimum of zero points on this problem.

On the pre-assessment, seven of the nine teachers demonstrated the basic skill of regrouping, and two of these teachers also demonstrated multiple perspectives of subtraction with regrouping. None of the teachers connected subtraction with regrouping to another mathematical topic, or mentioned longitudinal coherence. (See Fig. 1.1.1 in Appendix C, p. 74)

On the post-assessment, all nine teachers demonstrated the basic skill of regrouping, and seven of the teachers also used multiple perspectives in their explanation. Three of the teachers demonstrated longitudinal coherence by linking subtraction with regrouping to prior and/or subsequent topics. (See Fig. 1.1.2 in Appendix C, p. 74)

When comparing the scores on the pre- and post-assessments, six teachers’ scores rose one point, two teachers’ scores rose two points, and one teacher’s score did not change. (See Fig. 1.2.1 and Fig. 1.2.2 in Appendix C, p. 75)

Problem #2

<table>
<thead>
<tr>
<th>Explain the mathematical concepts you would review to help correct the mistake in the following problem:</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
</tr>
<tr>
<td>×645</td>
</tr>
<tr>
<td>615</td>
</tr>
<tr>
<td>492</td>
</tr>
<tr>
<td>738</td>
</tr>
<tr>
<td>1845</td>
</tr>
</tbody>
</table>

The rubric for problem #2 was similar to the rubric for problem #1 in the fact that one point was awarded for each of the four components of PUFM that the teacher
demonstrated. In place of subtraction with regrouping, problem #2 focused on the fundamental concept of multi-digit multiplication and place value. Teachers could earn a maximum of four points and a minimum of zero points on this problem.

Eight of the nine teachers in the study successfully demonstrated the basic skill of place value in multi-digit multiplication on the pre-assessment. One teacher used connectedness in their explanation, five teachers demonstrated multiple perspectives, and four teachers displayed longitudinal coherence. (See Fig. 2.1.1 in Appendix C, p. 77)

All nine teachers distinguished the basic skill of place-value in multi-digit multiplication on the post-assessment. Five teachers used connectedness, seven gave multiple perspectives, and three showed longitudinal coherence. (See Fig. 2.1.2 in Appendix C, p. 77)

When comparing the pre- and post-assessment scores for individual teachers, four teachers’ scores increased by one point, and one teacher’s score increased by two points. Four teachers’ scores remained the same. (See Fig. 2.2.1 and Fig. 2.2.2 in Appendix C, p. 78)

**Problem #3**

Write a word problem to represent $1 \frac{3}{4} \div \frac{1}{2}$. You need to devise a problem and solve the problem you wrote.

This problem required a more complex rubric, due to an overall weaker understanding of the concept of division by fractions. Many teachers struggled with the first step of being able to correctly solve and/or generate a word problem representing division by one half, and therefore could not demonstrate connectedness.
or longitudinal coherence. Rather than using a rubric that awarded one point for each component of PUFM, the rubric for problem three awarded one point for the correct answer, one point for a correct word problem, one point for demonstrating the correct procedural knowledge of division by fractions, and one point for demonstrating complete conceptual knowledge of the concept. Half points were awarded for teachers that came up with a mathematically correct word problem that had pedagogical errors, for example using halves of people. Half points were also awarded for teachers that demonstrated partial conceptual knowledge, for instance a response that correctly interpreted one and a half divided by one half, but did not correctly conceptualize one fourth divided by one half. Teachers could earn a maximum of four points on this problem, and a minimum of zero points.

On the pre-assessment only four teachers displayed the correct answer to the fractional division problem. Only two teachers came up with a fully correct word problem, and three additional teachers created a mathematically correct word problem with pedagogical errors. Four teachers demonstrated the correct procedural knowledge for division by fractions. None of the teachers demonstrated complete conceptual knowledge of division by fractions although two teachers demonstrated partial conceptual knowledge. (See Fig. 3.1.1 in Appendix C, p. 80)

Eight of the nine teachers were able to give a correct answer to the fractional division problem on the post-assessment. Three teachers created a fully correct word problem, and four teachers created a mathematically correct word problem with pedagogical errors. Seven teachers demonstrated correct procedural knowledge.
Three teachers showed complete conceptual knowledge while none of the teachers exhibited partial conceptual knowledge. (See Fig. 3.1.2 in Appendix C, p. 80)

All but two of the teachers’ scores increased from pre- to post-assessment. Three teachers’ scores increased by one point, one teacher’s score increased by one and a half points, and three teachers’ scores increased by two points. (See Fig. 3.2.1 and Fig. 3.2.2 in Appendix C, p. 81)

**Problem #4**

Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you the following picture to prove what she is doing. How would you respond to this student?

![Diagram](image)

P = 16 cm
A = 16 cm²

P = 24 cm
A = 32 cm²

This rubric was customized so as to better account for the wide array of responses. Some teachers were very specific in their answers, while others were ambiguous. Teachers that stated that the student was correct were awarded zero points. Teachers that responded with a purely instructional viewpoint were also awarded zero points. An example of this is, “I would congratulate the student for self-directed learning, and encourage them to continue investigating”. Although this may be an appropriate pedagogical response, there is no evidence of conceptual understanding of the mathematics. Teachers were awarded one point for prompting for or explaining a counterexample and were awarded another point for prompting the
student to use proof rather than relying on additional examples. A final point was awarded for a discussion on the different cases of the student’s proposition. For example, one case would be that as both the width and length increase, the area increases. Another case would be that as the width or length remains constant and the other dimension increases, the area increases. The third case is that as one dimension increases and the other dimension decreases, the area may not increase. Teachers could earn a half point with a partial discussion of these cases. Teachers could earn a maximum of three points on this problem, and a minimum of zero.

On the pre-assessment for this problem, two teachers stated that the student was correct. Only three teachers prompted for or gave a counterexample, and only one teacher discussed the need for proof. One teacher included a partial discussion of the different cases of the student’s proposition. (See Fig. 4.1.1 in Appendix C, p. 83)

On the post-assessment for this problem, there were no teachers that stated that the student was correct. Eight teachers either prompted for or gave a counterexample. Still only one teacher discussed the need for proof; however two teachers fully discussed the different cases involved in the student’s proposition. (See Fig. 4.1.2 in Appendix C, p. 83)

Six teachers’ scores increased by one point from pre- to post-assessment, and one teacher’s score increased by one half of a point. Two teacher’s scores remained the same. (See Fig. 4.2.1 and Fig. 4.2.2 in Appendix C, p. 84)

**Workshop Evaluation Survey**

At the end of the last workshop session, a survey was given to participants in order to evaluate the teachers’ perceptions of the success of the workshop. Teachers
were asked to evaluate how well the workshop met its goals, and how relevant the readings and workshop discussions were to their everyday practice. A four-point scale was used, and teachers were asked to give explanations for their ratings. For questions #1 through #3, the four point scale was defined with one denoting “The workshop did not meet its goal”, and four denoting “The workshop met its goal very well”. For questions #4 and #5, the four point scale was defined as one denoting “not relevant”, and four denoting “very relevant”.

**Question #1**

How well did this workshop meet its goal of deepening your understanding of subtraction with regrouping, multi-digit multiplication, division by fractions, and comparing area and perimeter?

The average rating for this question was a 3.111, with two teachers giving a rating of 2, four teachers giving a rating of 3, and three teachers giving a rating of 4. (See Fig. 5.1.1 in Appendix C, p. 86) None of the teachers felt that the workshop “did not meet its goal” of deepening understanding. Two teachers explained that they gained some insight into these mathematical topics, and four teachers referenced the benefit of learning different strategies for teaching. One teacher responded that the book study was offensive rather than informative.

**Question #2**

How well did this workshop meet its goal of improving your mathematical instructional methods?

The average rating for question #2 was a 3. One teacher gave a rating of 1, one teacher gave a rating of 2, four teachers gave a rating of 3, and three teachers gave a rating of 4. (See Fig. 5.1.2 in Appendix C, p. 86) Two teachers, the high
school teacher and a middle school teacher, responded that the strategies used were elementary in nature and therefore they may not be taking any instructional methods back to the classroom, however they commented that "the workshop helped me to refine my thoughts about my own methods and expectations" and "it did reinforce the connection to habits of mind". Two teachers mentioned specifically the concept of a "knowledge package" and its value in teaching math. One teacher stated, "I have already used the idea of a knowledge package when teaching multi-digit multiplication". Two other teachers referenced the usefulness of the multiple perspectives discussed in the workshop.

**Question #3**

How well did this workshop meet its goal of setting into motion a dedication to the deliberate study of the mathematics we teach?

The average rating for this question was a 3.556, with one teacher giving a rating of 2, two teachers giving a rating of 3, and six teachers giving a rating of 4. (See Fig. 5.1.3 in Appendix C, p. 87) Two teachers specified that the workshop was thought-provoking, with comments such as "This really got me thinking". Three teachers recognized the benefit of studying the math we teach, with comments such as "Even though I do not teach at the elementary level, this workshop has really illustrated how important it is to study how and what we teach", and "Math instructional practice and content needs to be included in staff development on an ongoing basis. There needs to be more dialogue K-12. This was a great beginning. Thank you."

Two teachers mentioned the benefit of the multiple perspectives. One of them wrote, "It's amazing to see how many methods we take for granted and allowed us to
analyze them more thoroughly.” Two teachers explained that time is an issue in meeting this goal of ongoing learning. They wrote, “The desire to learn more is there, however the time to dedicate to this study is scarce” and “I’ve always had this goal-time is always the big factor.”

**Question #4**

How relevant was the reading to your everyday practice?

The average rating for this question was a 3.111. On average teachers found the reading to be somewhat relevant to their practice. Two teachers gave a rating of 2, four teachers gave a rating of 3, and three teachers gave a rating of 4. (See Fig. 5.1.4 in Appendix C, p. 87) Three teachers described the relevancy of the reading to their practice by writing, “I deal with all concepts we studied in my fourth grade class”, “Any opportunity to explore insightful ways of teaching elementary math is relevant to a first grade teacher”, and “It opened my eyes to common misconceptions that students (and adults) have and ways to improve on them”. One teacher responded that the reading offended her, writing “I really was offended by most of what Ma wrote. Maybe I’m naïve, but I find it hard to believe that these U.S. teachers represent the majority of the population. Much effort was given to defending our practice.”

**Question #5**

How relevant were the discussions to your everyday practice?

The discussions seemed to be slightly more relevant to teachers than the reading, with an average rating for this question of 3.222. One middle-school teacher gave a rating of 1, with no explanation. Four teachers gave a rating of 3, and four
teachers gave a rating of 4. (See Fig. 5.1.5 in Appendix C, p. 88) Two teachers responded that the discussions deepened their understandings, and three teachers noted that the discussions helped to strengthen their teaching practices. These comments include, "I will definitely include some of these deep understandings in my discussions and encourage students to think of different ways of solving problems, not just one set way", and "The discussions did help me think about my teaching practices even if it's not the same content". One elementary teacher explained how the discussions solidify curriculum flow and longitudinal coherence, writing "Aside from the relevant topic, it was beneficial to see how my first grade teaching will lead to success or trouble in future years. It's a huge responsibility!"
Chapter 6

Data Analysis
Pre- and Post-Assessment

Problem #1

The null hypothesis for this particular problem was that teachers would not exhibit a significantly deeper understanding of subtraction with regrouping after having gone through the workshop sessions and reading Ma’s book. The alternative hypothesis was that teachers would exhibit a deeper understanding of subtraction with regrouping. This “deeper understanding” was measured on the pre- and post-assessment using the rubric described in the previous chapter. Using a t-distribution for significance testing, with a .01 level of significance, it is required that the calculated t-score falls outside the critical values -3.250 and 3.250. The t-score is calculated to be 5.547 for problem #1, meaning that we accept the alternative hypothesis and conclude that there is a significant difference in the pre- and post-assessment scores. (See Fig. 1.3.1 in Appendix C, p. 76) The participants exhibited a deeper understanding of subtraction with regrouping after having gone through the workshop sessions and reading Ma’s book.

For this particular problem the average score increased from 1 point per teacher to 2.111 points per teacher. (See Fig. 1.1.3 in Appendix C, p. 74) All but one teacher’s score increased by at least one point. The most notable improvement was the ability for seven teachers to respond with multiple perspectives in the post-assessment, when only two teachers used multiple perspectives in the pre-assessment. This may be partially due to the fact that workshop session 1 asked teachers to share different methods of solving a subtraction with regrouping problem, and furthermore, during workshop session 2, teachers were asked to analyze these different methods.
Through discussions teachers were enlightened to alternative methods of conceptualizing subtraction with regrouping, and then also weighed the advantages and disadvantages of each method. Specifically the case of subtraction with minuends less than 20 forced teachers to think of alternative methods to "borrowing". Interestingly, the middle and high school teachers were the ones who were most unfamiliar with the alternative methods that could apply to this case of subtraction with regrouping. The problem $15 - 7$, for example, was strictly an "arithmetic fact" for the middle and high school teachers, and they expected students to simply start at 15 and count down 7. They were not familiar with alternative methods, such as mentally computing $15 - 10$, and then adding 3 to the answer, or mentally subtracting $15 - 5$, and then subtracting 2 more from the answer. The elementary school teachers referred to these methods as "shifting values". These "non-standard" methods of regrouping were unfamiliar to the middle and high school teachers. The two teachers that scored 0 points on the pre-assessment were both middle school teachers, whose responses were procedural rather than conceptual. One of these teachers wrote the following:

"We must look at the ones column first. Since seven is greater than 2, we must borrow a group of ten from the tens column. We will make the 2 into 12. Then, subtract 7 from 12. Write five in the ones column. When we borrow one group of 10, we have four groups of ten left. Subtract two from four and there are two left. The answer is 25."

This response is procedural in nature, and dangerous in that she says, we “make the 2 into 12”, without reference to the fact that one tens is equivalent to ten ones. She does speak of groups of tens, however, which is the beginning of a conceptualization of place value.
The elementary school teachers shared methods such as “counting on”, for example seeing how many numbers a student would count, starting at 7, to get to 15, and “shifting values”, mentioned above. One elementary school teacher’s response to problem #1 was as follows:

“One way to think about subtraction is to think about the distance or the difference between the two numbers. How far apart are 52 and 27? If we start at 27 on the hundreds chart, how many “jumps” to 30? Then to 50? Then to 52?”

Although this is a conceptual response, it raises a question worthy of discussion: does this teach the concept of regrouping? In small groups teachers discussed not only the advantages and disadvantages to this method and other methods, but also identified specifically what concepts each method teaches students. While something like the “counting on” method is useful for some subtraction with regrouping problems, when students are confronted with large numbers will this method be sufficient?

The traditional method of "borrowing" remained important to all teachers; although it was renamed "decomposing a higher valued number" so as to better convey the concept of regrouping and place value. However this method does not address subtraction problems that involve a minuend less than 20. It became evident to all teachers that teaching students various methods of regrouping is necessary in order for them to fully conceptualize and become efficient in subtracting.

All teachers seemed to gain something from the discussions on subtraction with regrouping. For particularly the middle and high school teachers, the discussions helped to enlighten them to the different conceptualizations of regrouping, such as "counting on" or "shifting values". The elementary school teachers also seemed to expand their perspectives throughout the discussions on
different methods. Many of them commented that using the language "decomposing" when teaching students about place value and subtracting made much more sense than using the term "borrowing". This terminology highlights the concept rather than simply the procedure.

**Problem #2**

The null hypothesis for this problem, which tested multi-digit multiplication and the concept of place value, was that there would be no difference between scores on the pre- and post-assessment. The alternative hypothesis was that there would be a significant difference between scores. Again using a level of significance of .01 which gives critical t-values of -3.250 and 3.250, the calculated t-score must fall outside this range in order to conclude that there is a significant difference in scores. The calculated t-score is 2.828, which does not fall outside the given range. (See Fig. 2.3.1 in Appendix C, p. 79) This means that the alternative hypothesis is rejected and we conclude that there is not a significant difference in performance.

The average number of points scored on the pre-assessment for this problem was 2, and the average number of points scored on this problem on the post-assessment was 2.667, an increase of only 2/3 of a point. (See Fig. 2.1.3 in Appendix C, p. 77) Eight of nine teachers came into the workshop with an understanding of the basic skills used in multi-digit multiplication. The one teacher that did not exhibit a conceptual understanding of place value on the pre-assessment was a middle school teacher, who wrote:

"Each time you multiply a number, you have to start your answer right under that number so we put a zero under the numbers we do not use. We call them place holders. When we multiply 123 by 4, the first number of the answer begins in the column right under the 4."
This response that is strictly procedural and although it is not incorrect, it lacks the components of PUFM. This teacher’s explanation does not promote conceptual thinking and does not explain the meaning of place value.

Most teachers were able to describe the multi-digit multiplication problem from multiple perspectives, such as using the distributive property, matrices, and adding partial products. Two elementary school teachers’ responses also prompted the student to determine the reasonableness of their answer, which promotes longitudinal coherence and a connection to estimation. The following is one of these teachers’ responses:

“I would have the student decompose the 645 to 600 + 40 + 5 and multiply each part separately. Then connect their answers to the algorithm. Also- looking at 738 line, Does it make sense that 600 x 100 = about 700?”

Another elementary school teacher identified the mathematical concepts they would review as, “1. Place value: What does the ‘4’ mean? The ‘6’? What are you multiplying by? 2. What is the meaning of multiplication?” This response prompts a conceptual understanding of place value, the basic skill in this problem, and also links the concept of multiplication to its meaning of repeated addition, which students will relate to prior knowledge.

While most of the discussion for this problem involved teachers sharing different methods that they felt would reinforce the concept of place value, a debate did ensue on the importance of using 0 as a place holder. The following are examples of the slightly different methods:
Using 0s as place holders:

```
  123
x645
  615
 4920
 73800
 79335
```

Omitting 0s:

```
  123
x645
  615
  492
  738
  79335
```

Some teachers argued that using 0 is extremely important and fundamental to multi-digit multiplication, while others believed that the 0s simply represented place holders and whether the symbol of 0 is written or not is arbitrary as long as the “place is held”. Most teachers eventually came to an agreement that after students conceptualized the meanings of place value and place holders, the actual writing of the 0s could be omitted. Once students understand that a 2 placed in the second column has a different meaning than a 2 placed in the first column, not simply due to procedure or organization, but because places have meaning, the actual use of 0 is arbitrary. Students would know that a 0 placed in the ones column would have the same meaning as nothing being placed in the ones column and that a number in the tens column has meaning and cannot not simply “slide over” to the ones column.

Most teachers came into the workshop already having developed a basic understanding of the meaning of place value in multi-digit multiplication. Furthermore, over half of the participants recognized multiple perspectives of this topic on the pre-assessment. This prior knowledge was the major reason that there
was not a significant difference in scores on the pre- and post-assessment. A quick discussion of the reading on multi-digit multiplication seemed to solidify conceptual understanding and help some teachers gain multiple perspectives. Five teachers improved in their attainment of PUFM, primarily by describing connections between this topic and other topics and by gaining multiple perspectives.

Problem #3

Like problems #1 and #2, the null hypothesis for problem #3 was that there would be no significant difference between the pre- and post-assessment scores for this problem, which deals with fractional division. The alternative hypothesis was that there would be a significant difference in teacher performance on the two assessments. Again using a significance level of .01 for the t-distribution hypothesis test, the calculated t-score must fall outside the critical values of -3.250 and 3.250. The calculated t-score for this problem is 4.427. (See Fig. 3.3.1 in Appendix C, p. 82) Therefore, the alternative hypothesis is accepted meaning that there was a significant difference in scores for problem #3.

The average score per teacher for this problem increased from 1.389 to 2.556. (See Fig. 3.1.3 in Appendix C, p. 80) The number of teachers that correctly answered the fractional division problem increased from four to eight. Seven teachers were able to demonstrate correct procedural knowledge on the post-assessment, whereas only four teachers knew the procedure for fractional division on the pre-assessment.

The statistics mentioned above are evidence of improvement in mathematical content knowledge, a goal of the workshop. However, although the number of teachers that demonstrated complete conceptual knowledge increased from 0 to 3, all
three of these teachers were middle or high school teachers. Also, all correct word
problems on the post-assessment were completed by middle or high school teachers.
The elementary school teachers seemed to have a much weaker understanding of
fractional division. For example, a first grade elementary school teacher wrote:

"You are packing up left-over pizza from the party and want to equally
split it with your friend. There is only 1 3/4 pizzas left of the 4 that the
party started with. If each pizza had 8 slices to begin with, how many
pieces will you and your friend each take home? 1/2 of 1 whole = 4
times 1/2 of 3/4 is 3 pieces **Answer = each of you will take 7
pieces of pizza home."

This response demonstrates many gaps in understanding. First of all, the answer is
intended to be in terms of whole pizzas, making the answer 7/8 of a pizza, this teacher
would be demonstrating 1 3/4 times 1/2, not 1 3/4 divided by 1/2. Because this teacher did
not know the procedure for fractional division, she could not check her answer. A
fourth grade teacher responded with the following word problem:

"When baking cookies, Suzy needed to add 1 3/4 cups of flour to bake
12 cookies. She needed to bake 24 cookies, so how much flour would
she need? 1 3/4 = 7/4 + 1/2 = 7/4 × 2 = 14/4 = 3 2/4 = 3 1/2 cups"

Although this teacher came up with the correct answer by using the correct procedure,
the word problem she wrote represents 1 3/4 times 2, not 1 3/4 divided by 1/2. It is the
concept of fractional division that is missing.

The following response was given by a middle school teacher and represents
partial conceptual understanding:

"I have 1 3/4 lbs of candy and I want to give each person at the party 1/2
log. How many people can I serve? How much candy is left over? 3
people, 1/4 lb left over". (See Response 1 in Appendix E, p. 121)
This person has come up with a mathematically correct word problem that represents 1¾ divided by ½, however the answer to the problem is incomplete. Although the teacher correctly computes 1½ divided by ½, she does not interpret the meaning of ¼ divided by ½. This may be because the word problem she has created leads to an answer of 3 and a half people, which practically does not make sense. Although this teacher demonstrates a partial conceptual understanding of fractional division, her response is lacking a final conclusion.

The high school teacher gave a response that demonstrated complete conceptual knowledge:

"You have one full snack bar and ¾ of another. How many pieces will you have if you cut lengths equal to half of the full snack bar? \( \frac{1\frac{3}{4}}{\frac{1}{2}} = \frac{7}{4} \)

\[ \frac{7}{2} \div \frac{1}{2} = \frac{7}{4} \cdot 2 = \frac{7}{2} = 3.5 \] 3 groups of ½, 1 of ¼ (¼ is ½ of ½) so 3.5"
(See Response 2 in Appendix E, p. 121)

Although the context of this problem may be slightly confusing for students, the word problem is mathematically representative of 1¾ divided by ½, and her response demonstrates the concept of dividing by one half.

This concept seemed to have the most variety in prior knowledge. Some teachers were able to describe partial conceptual understandings on the pre-assessment, while other teachers could not perform the correct procedure for fractional division. Overall, the middle and high school teachers were more advanced in their understanding of this topic. Two elementary school teachers did not demonstrate correct procedural knowledge even on the post-assessment. While workshop session 4 did address the algorithm for fractional division, different models of word problems, and student misconceptions, it could be that the middle and high
school teachers were driving these discussions, as they were more advanced with the content and prepared to analyze and converse. Teachers that felt uncomfortable with the basic mathematical operation of fractional division may have felt reluctant to participate in discussions about word problems and student misconceptions. Seven of the nine teachers in the study scored higher on the post-assessment than on the pre-assessment, however for the elementary school teachers, the progress resulted from an improvement in the correct procedural knowledge and correct answer. While this is an important step towards developing an understanding of the concept, to have no elementary school teacher demonstrating full or partial conceptual knowledge of fractional division on the post-assessment is a red flag for this study. Although the hypothesis testing showed that there is a significant difference in the scores on the pre- and post-assessment, which shows that significant improvements were made, perhaps just as importantly are the improvements made in the category of conceptual knowledge. The workshop may not have met its goal of deepening the understanding of this concept for all teachers.

**Problem #4**

The null hypothesis for this problem was that there would be no difference in scores on the pre- and post-assessment. The alternative hypothesis was that there would be a significant difference in scores. This problem tested the concept of perimeter and area, as well as the concept of proof. The calculated t-score is 4.914 which falls outside of the range -3.250 and 3.250, meaning that the alternative hypothesis is accepted for a .01 level of significance. (See Fig. 4.3.1 in Appendix C, p. 85) There is a significant difference in scores for problem #4.
The average points per person rose from .5 to 1.222 for this problem. (See Fig. 4.1.3 in Appendix C, p. 83) On the post-assessment there was significant improvement in recognizing counterexamples, whether it was by teacher prompting or informing the student. Also on the post-assessment, no teacher concluded that the student was correct, whereas two teachers agreed with the student on the pre-assessment. For these two teachers, a misconception was corrected and their mathematical content knowledge was improved.

This question should be reworded so as to better prompt the teachers for information. Since the question is open-ended one teacher simply wrote, "I would congratulate the student for self-directed learning, and encourage them to continue investigating." This is an appropriate response if the teacher is aware of the counterexamples and is prepared to guide the student if necessary. If the teacher is responding this way to the student because they themselves do not know the answer, this could lead to a dangerous misconception. While some teachers explicitly stated that they thought the student was correct or incorrect, teachers that answered the question from a purely instructional viewpoint were not awarded any points unless they were more specific as to how they would prompt for counterexamples or discuss the different cases of the situation. Teachers that simply asked the student for more examples were awarded zero points, again because of ambiguity of content knowledge. Many elementary school teachers described instructional methods and the content knowledge of the teacher was left uncertain. This may or may not be because the teachers are weak in their content knowledge. However, it is concerning to think that the instructional methods would be driving the content, rather than a
teacher's deep understanding of the content knowledge guiding the choice for instructional method. This question would have been more effective as an interview question, where teachers could be prompted if their answers were ambiguous.

A non-ambiguous response was given by another elementary school teacher who has been teaching for over 20 years:

"I would ask her if this works for all closed figures. I would ask her to try convex and concave figures with the same perimeter (use graph paper) Also, think of other factor pairs of 32. Should a 1 x 32 rectangle have a perimeter of 24?" (See Response 3 in Appendix E, p. 121)

This response uses instructional methods of inquiry and questioning, however it is evident that the questions posed will lead the student to recognizing counterexamples. The teacher is guiding toward correct conceptual understandings.

The ambiguity this problem allows for makes it difficult to assess teachers. However, this is not the only trouble with this problem. This assessment question is also problematic because it tests two different conceptual ideas- area and perimeter, as well as the meaning of proof. Not all teachers acknowledged the need to address the meaning of proof with students. For example, an elementary school teacher responded as follows:

"I would applaud her self-directed learning and then ask the entire class to find examples to prove or disprove her theory. We would organize the data into a list (an organized list) and then draw conclusions based on the findings."

Not only is this example of an ambiguous response and unclear as to whether or not the teacher will be able to guide the student toward recognizing their misconception, but this response also implies that numerous examples are sufficient to prove a
theory. Although analyzing patterns is a valuable problem-solving strategy, it is important that students understand the meaning of proof.

A middle school teacher, however, gave the following response:

"I would ask if she has tried lots of other examples to help prove her point. I might say that if she and I both wear glasses, does that prove that all people wear glasses? I would then get some tiles or graph paper and draw lots of rectangles, getting the perimeter and area of each. Once she finds one that doesn't agree with her discovery, I would probably talk about what kind of steps are necessary to formulate a "discovery". The talk could also include other shapes, as her statement didn't specify rectangles."

This illustrates how, with teacher guidance, even a young student can understand the basic meaning of proof.

As a whole, the elementary school teachers gave more ambiguous, pedagogically driven responses to this problem. The middle and high school teachers were more likely to investigate the mathematical inquiry themselves, and also more likely to reinforce the concept of proof. These findings raise the question, "Where should students be in their development of the concept of proof in elementary school?" From the inconsistent and vague responses from many elementary school teachers it seems as though this matter may need more discussion.

**Workshop Evaluation Survey**

The results from the evaluation survey give valuable information about the success of this research. All three questions that asked teachers to rate how well the workshop met its goals received average scores of 3 or better.

The lowest score, averaging 3 points on the 4-point scale was given for the second goal, which was, "The workshop will improve your mathematical instructional methods." This was due to the fact that specifically the middle and high
school teachers, as well as the reading teacher, felt that the specific content would not be applicable to them. However, many of these teachers did comment that the new methods learned would help them to support the topics they teach.

The highest score, averaging 3.556 out of 4, was given for the third goal, “The workshop will set into motion a dedication to the deliberate study of the mathematics we teach.” Because Ma’s research has shown that teachers attain a Profound Understanding of Fundamental Mathematics while they are teaching, due to constant analysis of mathematics and instruction, this goal is the most important when looking at the “big picture” of this research. No teacher felt that the workshop did not meet this goal. Teachers of all levels commented that studying, discussing, and sharing mathematical ideas is extremely beneficial in their careers.

The first goal, which was “The workshop will deepend understanding of four fundamental mathematical concepts” received an average score of 3.111. Like the responses for the second goal, some middle and high school teachers in particularly commented that although they learned new strategies, the actual content was not new to them.

The last two questions on the evaluation survey asked teachers to determine how relevant the readings and discussion were to their everyday practice. The results found that on average the book earned a relevancy of 3.111 out of 4 and the discussions earned a score of 3.222 out of 4. The common responses of the middle and high school teachers were that the content was not curriculum-relevant to them. Even with these comments, however, most teachers felt that the reading and discussions were somewhat relevant to them.
Furthermore, when asked about the relevancy of the reading, some teachers expressed that they were offended by Ma’s book, feeling that it misrepresented United States elementary school teachers. Due to this common reaction, the last workshop session presented and discussed the differences in teaching circumstances in China and the U.S., according to Ma’s book. Ma writes that many of the teachers in China are math specialists who teach three or four 45 minute classes a day, with the rest of the time spent planning and studying mathematics. (Ma, 1999, p. 129)

The elementary school teachers in this workshop expressed that they can spend only about 20 minutes a day studying math, and their college math requirements consisted of a maximum of 12 credit hours of mathematics. At this rate, a Chinese elementary school teacher that did not go to college would surpass a U.S. elementary school teacher that did go to college in his or her study of mathematics after only 60 days of teaching. (See Power Point 5 in Appendix D, p. 117) This is consistent with Ma’s conclusion that PUFM is attained while teaching. This discussion helped the teachers in the workshop to feel less personally attacked and enlightened them to the larger issues that must be addressed when attempting to improve teacher content knowledge of mathematics.
Chapter 7

Reflections
Throughout this action research my perspective on the learning of mathematics has widened drastically. Teaching and learning mathematics is not a task that can be mastered once-and-for-all, but instead it is an endless, living understanding that requires constant cultivation and development. Nor does the teaching and learning of mathematics have a starting or ending point. It is essential that every mathematics teacher can conceptualize the mathematics before and after what he or she teaches so as to guide the students toward a seamless understanding of mathematics.

Throughout the workshop and book study, I was able to observe teachers of various grade levels each develop their profound understanding of fundamental mathematics. Some middle and high school teachers needed to be reminded of the concepts behind their procedures and algorithms, while some elementary school teachers learned why appropriate terminology and attention to details such as the meaning of proof are crucial to a student’s mathematical future.

The next step in this action research is to collaborate with a K-12 curriculum coordinator and facilitate another workshop, with the following changes to be made. First of all, with the curriculum coordinator’s help the workshop would hope to include many more participants. As lack of time has been identified as a major deterrent in achieving the goals of this research, I would work with the curriculum coordinator and policy-makers to help make time for this workshop to occur. The workshop would still include teachers of all levels, as this seemed to add valuable perspectives to the discussions. Secondly, the workshop would not be based around Ma’s book, as her research was offensive to many elementary school teachers.
Instead, the workshop would contain the same conceptual ideas, with examples from Ma's book and other research. Thirdly, the pre- and post-assessment would be reworded so as to avoid ambiguous responses. Also, if possible, participants would be interviewed so as to better assess where they are in their conceptual understanding. This accurate assessment information would help to modify the workshop to the needs of the teachers, making it relevant and beneficial for all participants. Lastly, the workshop would spend more time teaching and investigating fractional division and the meaning of proof.

This research has impacted my practice in a variety of ways. First of all, I have made it a career goal of mine to learn the curriculums of the courses before and after the courses that I teach. On a smaller scale, I have begun to approach every lesson with the mindset that the concept being taught is a continuation of something the student already knows. I not only recognize their prior knowledge, but consistently rely on it in order to link new concepts to current knowledge. Another impact this research has had on my professional practice is that I do have a deeper understanding of four fundamental mathematical concepts. I did not realize that my understanding was "incomplete" until I read Ma's book and facilitated the workshop sessions in which I learned alternative ways to conceptualize the given problems. A third impact of this research is that I am now interested in the K-12 mathematics curriculum and would like to relay what I've learned and the implications of Ma's study to curriculum coordinators and policy makers. The benefit of professional development opportunities in which teachers deepen their mathematical understandings has proven too effective and significant to be ignored.
Chapter 8

References


Chapter 9

Appendices
Appendix A
Pre- and Post-Assessment

Knowing and Teaching Elementary Mathematics

1. How would you explain to a group of second graders how to solve the following problem?

\[
\begin{array}{c}
52 \\
-27 \\
\end{array}
\]

2. Explain the mathematical concepts you would review to help correct the mistake in the following problem:

\[
\begin{array}{c}
123 \\
\times 645 \\
615 \\
492 \\
738 \\
1845 \\
\end{array}
\]

Liping Ma (1999) Knowing and Teaching Elementary Mathematics
3. Write a word problem to represent $1 \frac{3}{4} \div \frac{1}{2}$. You need to devise a problem and solve the problem you wrote.

4. Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you the following picture to prove what she is doing. How would you respond to this student?

Liping Ma (1999) *Knowing and Teaching Elementary Mathematics*
Appendix B

Workshop Evaluation Survey

Please answer the following questions on a scale of one to four, with one being “The workshop did not meet its goal”, and four being “The workshop met its goal very well”. Then, please explain your choice in terms of your own experience. Thank you!

1. How well did this workshop meet its goal of deepening your understanding of subtraction with regrouping, multi-digit multiplication, division by fractions, and comparing area and perimeter?

   1  2  3  4

   Please Explain:

   

2. How well did this workshop meet its goal of improving your mathematical instructional methods?

   1  2  3  4

   Please Explain:

   

3. How well did this workshop meet its goals of setting into motion a dedication to the deliberate study of the mathematics we teach?

   1  2  3  4

   Please Explain:
Please answer the following question on a scale of one to four with one being “not relevant” and four being “very relevant”.

4. How relevant was the reading to your everyday practice?

1 2 3 4

Please Explain:

5. How relevant were the discussions to your everyday practice?

1 2 3 4

Please Explain:
Appendix C
Data Collection and Hypothesis Testing

**Fig. 1.1.1 Pre-Assessment Problem #1**

<table>
<thead>
<tr>
<th>Distinguish Basic Skill (1 pt)</th>
<th>Connectedness (1 pt)</th>
<th>Multiple Perspectives (1 pt)</th>
<th>Longitudinal Coherence (1 pt)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Total number of points earned: 9  
Average points per person: 1

**Fig. 1.1.2 Post-Assessment Problem #1**

<table>
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<tr>
<th>Distinguish Basic Skill (1 pt)</th>
<th>Connectedness (1 pt)</th>
<th>Multiple Perspectives (1 pt)</th>
<th>Longitudinal Coherence (1 pt)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>0</td>
<td>7</td>
<td>3</td>
</tr>
</tbody>
</table>

Total number of points earned: 19  
Average points per person: 2.111

**Fig. 1.1.3 Comparing Averages on Pre- and Post-Assessment for Problem #1**

![Bar chart comparing pre-assessment and post-assessment averages](image-url)
Fig. 1.2.1 Table of Individual Teacher Scores on Problem #1

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Pre-Assessment Points Earned</th>
<th>Post-Assessment Points Earned</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
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</tr>
<tr>
<td>B</td>
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</tr>
<tr>
<td>C</td>
<td>1</td>
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<td>D</td>
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<td>2</td>
</tr>
<tr>
<td>I</td>
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Fig. 1.2.2 Graph of Individual Teacher Scores on Problem #1
Figure 1.3.1 Hypothesis Testing for Problem #1

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Pre-Assessment Points Earned</th>
<th>Post-Assessment Points Earned</th>
<th>Difference</th>
<th>Difference Squared</th>
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</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>2</td>
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<td>C</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>1</td>
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<tr>
<td>D</td>
<td>0</td>
<td>2</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>E</td>
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<td>I</td>
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</tr>
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</table>

\[
\bar{x}_1 = 1 \quad \bar{x}_2 = 2.111 \quad \sum \text{diff} = -10 \quad \sum \text{diff}^2 = 14
\]

\[
N = 9 \quad d.f. = 8 \quad t = \frac{|1 - 2.111|}{\sqrt{\frac{9(14) - (-10)^2}{81(8)}}} = \frac{1.111}{\sqrt{\frac{26}{648}}} = \frac{1.111}{.200} = 5.547
\]
### Fig. 2.1.1 Pre-Assessment Problem #2

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<th>Multiple Perspectives</th>
<th>Longitudinal Coherence</th>
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<tr>
<td>8</td>
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<td>4</td>
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Total number of points earned: 18  
Average points per person: 2

### Fig. 2.1.2 Post-Assessment Problem #2

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</tr>
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<td>3</td>
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</tbody>
</table>

Total number of points earned: 24  
Average points per person: 2.667

### Fig. 2.1.3 Comparing Averages on Pre- and Post-Assessment for Problem #2

![Bar Chart Comparing Averages](image)
Fig. 2.2.1 Table of Individual Teacher Scores on Problem #2

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<td>I</td>
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Fig. 2.2.2 Graph of Individual Teacher Scores on Problem #2
Figure 2.3.1 Hypothesis Testing for Problem #2

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Pre-Assessment Points Earned</th>
<th>Post-Assessment Points Earned</th>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>H</td>
<td>0</td>
<td>2</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>I</td>
<td>2</td>
<td>3</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

\[
\bar{x_1} = 2 \quad \bar{x_2} = 2.667 \quad \sum \text{diff} = -6 \quad \sum \text{diff}^2 = 8
\]

\[
N = 9 \quad \text{d.f.} = 8
\]

\[
t = \frac{|2 - 2.667|}{\sqrt{\frac{9(8) - (-6)^2}{81(8)}}} = \frac{.667}{.235} = 2.828
\]
### Fig. 3.1.1 Pre-Assessment Problem #3

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Total number of points earned: 12.5  
Average points per person: 1.389

---

### Fig. 3.1.2 Post-Assessment Problem #3

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

Total number of points earned: 23  
Average points per person: 2.556

---

### Fig. 3.1.3 Comparing Averages on Pre- and Post-Assessment for Problem #3

![Bar chart comparing pre-assessment and post-assessment averages for Problem #3](chart.png)
**Fig. 3.2.1 Table of Individual Teacher Scores on Problem #3**

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Pre-Assessment Points Earned</th>
<th>Post-Assessment Points Earned</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>3.5</td>
</tr>
<tr>
<td>D</td>
<td>.5</td>
<td>2.5</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>F</td>
<td>1.5</td>
<td>2.5</td>
</tr>
<tr>
<td>G</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>H</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>I</td>
<td>.5</td>
<td>2.5</td>
</tr>
</tbody>
</table>

**Fig. 3.2.2 Graph of Individual Teacher Scores on Problem #3**
Figure 3.3.1 Hypothesis Testing for Problem #3

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Pre-Assessment Points Earned</th>
<th>Post-Assessment Points Earned</th>
<th>Difference</th>
<th>Difference Squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>3</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>3.5</td>
<td>-1.5</td>
<td>2.25</td>
</tr>
<tr>
<td>D</td>
<td>.5</td>
<td>2.5</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>3</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>F</td>
<td>1.5</td>
<td>2.5</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>G</td>
<td>3</td>
<td>4</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>H</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>I</td>
<td>.5</td>
<td>2.5</td>
<td>-2</td>
<td>4</td>
</tr>
</tbody>
</table>

\[ \bar{x}_1 = 1.389 \quad \bar{x}_2 = 2.556 \quad \sum \text{diff} = -10.5 \quad \sum \text{diff}^2 = 17.25 \]

\[ t = \frac{1.389 - 2.556}{\sqrt{\frac{9(17.25) - (-10.5)^2}{81(8)}}} = \frac{1.167}{1.167} = 1.167 = 4.427 \]

\[ N = 9 \quad d.f. = 8 \]

"Accept Ho"
Fig. 4.1.1 Pre-Assessment Problem #4

<table>
<thead>
<tr>
<th>Prompt for or Give Counterexample (1 pt)</th>
<th>Prompts for Proof (rather than simply examples) (1 pt)</th>
<th>Discusses Cases (1 pt)</th>
<th>Partial Discussion of Cases (.5 pt)</th>
<th>States that Student is Correct (0 pt)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Total number of points earned: 4.5 Average points per person: .5

Fig. 4.1.2 Post-Assessment Problem #4

<table>
<thead>
<tr>
<th>Prompt for or Give Counterexample (1 pt)</th>
<th>Prompts for Proof (rather than simply examples) (1 pt)</th>
<th>Discusses Cases (1 pt)</th>
<th>Partial Discussion of Cases (.5 pt)</th>
<th>States that Student is Correct (0 pt)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Total number of points earned: 11 Average points per person: 1.222

Fig. 4.1.3 Comparing Averages on Pre- and Post-Assessment for Problem #4

![Diagram](image)
Fig. 4.2.1 Table of Individual Teacher Scores on Problem #4

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Pre-Assessment Points Earned</th>
<th>Post-Assessment Points Earned</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>D</td>
<td>.5</td>
<td>1</td>
</tr>
<tr>
<td>E</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>F</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>G</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>H</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>I</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Fig. 4.2.2 Graph of Individual Teacher Scores on Problem #4
**Figure 4.3.1 Hypothesis Testing for Problem #4**

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Pre-Assessment Points Earned</th>
<th>Post-Assessment Points Earned</th>
<th>Difference</th>
<th>Difference Squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>0.5</td>
<td>1</td>
<td>-0.5</td>
<td>0.25</td>
</tr>
<tr>
<td>E</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>F</td>
<td>2</td>
<td>3</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>G</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>H</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>I</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

\[
\bar{x}_1 = 0.5 \quad \bar{x}_2 = 1.222 \quad \sum \text{diff} = -6.5 \quad \sum \text{diff}^2 = 6.25
\]

\[
N = 9 \\
d.f. = 8
\]

\[
t = \frac{|0.5 - 1.222|}{\sqrt{\frac{9(6.25) - (-6.5)^2}{81(8)}}} = \frac{0.722}{0.147} = 4.914
\]

**Diagram:**
- Accept \(H_0\)
- Critical value: \(t = 4.914\)
5.1.1 Survey Question #1

How well did the workshop meet its goal of deepening your understanding of mathematics?

5.1.2 Survey Question #2

How well did the workshop meet its goal of improving your mathematical instructional methods?
5.1.3 Survey Question #3

How well did the workshop meet its goal of setting into motion a dedication to the deliberate study of the mathematics you teach?

5.1.4 Survey Question #4

How relevant was the reading to your everyday practice?
5.1.5 Survey Question #5

How relevant were the discussions to your everyday practice?
## Appendix D
### Workshop Resources

## Facilitating Guide Session #1

<table>
<thead>
<tr>
<th>Activity/Discussion</th>
<th>Time Frame</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>~5-10 min</td>
</tr>
<tr>
<td>• Ask teachers to take turns sharing names, positions, and reason for taking this workshop.</td>
<td></td>
</tr>
<tr>
<td>• Give books, sign consent forms.</td>
<td></td>
</tr>
<tr>
<td>(Power Point slides 1-2)</td>
<td></td>
</tr>
<tr>
<td>Goals of workshop:</td>
<td>~3 min</td>
</tr>
<tr>
<td>• Enrich content understanding of four math concepts</td>
<td></td>
</tr>
<tr>
<td>• Improve mathematical instructional methods</td>
<td></td>
</tr>
<tr>
<td>• Set in motion a dedication to deliberate study of math</td>
<td></td>
</tr>
<tr>
<td>(Power Point slide 3)</td>
<td></td>
</tr>
<tr>
<td>Background</td>
<td>~7 min</td>
</tr>
<tr>
<td>• “Missing Paradigm” of math content</td>
<td></td>
</tr>
<tr>
<td>• Discussion: Where does Penfield fall on spectrum?</td>
<td></td>
</tr>
<tr>
<td>• Common assumptions</td>
<td></td>
</tr>
<tr>
<td>(Power Point slides 4-7)</td>
<td></td>
</tr>
<tr>
<td>Introduction to Liping Ma’s Study</td>
<td>~7 min</td>
</tr>
<tr>
<td>• “If a teacher’s own knowledge of the mathematics… is limited to procedures, how could we expect his or her classroom to have a tradition of inquiry mathematics?” (Ma, 1999, p. 153)</td>
<td></td>
</tr>
<tr>
<td>• Procedure of her study</td>
<td></td>
</tr>
<tr>
<td>• Differences in culture: “Behind the Scenes” (Ma, 1999, p. 129)</td>
<td></td>
</tr>
<tr>
<td>(Power Point slides 8-10)</td>
<td></td>
</tr>
<tr>
<td>Foreword #1, 2, 3</td>
<td>~5 min</td>
</tr>
<tr>
<td>(Power Point slides 11-13)</td>
<td></td>
</tr>
<tr>
<td>Pre-assessment</td>
<td>~20-25 min</td>
</tr>
<tr>
<td>(Power Point slide 14)</td>
<td></td>
</tr>
<tr>
<td>Subtraction Activity:</td>
<td>~30 min</td>
</tr>
<tr>
<td>• Solve the Problem $\frac{63}{-29}$</td>
<td></td>
</tr>
<tr>
<td>• In groups, share and discuss strategies used</td>
<td></td>
</tr>
<tr>
<td>• Consider the problem $\frac{15}{-7}$</td>
<td></td>
</tr>
<tr>
<td>• Are the strategies discussed in the previous problem applicable to this problem? What other methods could be used to solve this problem?</td>
<td></td>
</tr>
<tr>
<td>(Power Point slides 15-16)</td>
<td></td>
</tr>
<tr>
<td>Give assignment for next week: Guiding Questions 1</td>
<td>~3 min</td>
</tr>
</tbody>
</table>

Attachment 1: Power Point Presentation 1
Attachment 2: Guiding Questions for Chapter #1
Appendix A: Pre-assessment
Power Point Presentation 1

Deepening Fundamental Mathematics

A BOOK STUDY

Introduction

- Name
- Position, number of years at Penfield
- Reason for taking this workshop

Goals of Workshop

- Enrich content understanding of four mathematical topics
- Improve mathematical instructional methods
- Set in motion a dedication to the deliberate study of the math we teach

Background

- 1989-NCTM shifted focus from WHAT to teach to HOW to teach
- Teacher Certification requires more classes on pedagogy than on math content
- Recent "Math Wars" argued instructional methods with little mention of content

"Missing Paradigm"

Where did the content go?

Deep Understanding of Content

Where does Penfield fall?

Pedagogy

Content
Background

Although the study of pedagogy is an essential part of successful teaching, why has the knowledge of content been neglected?

- Assumption #1: Math is not hard
- Assumption #2: Teachers already learned content well enough through their own upbringing

Liping Ma's Study

- Interviewed 29 elementary teachers from the United States, 72 teachers from China
- U.S. Teachers: 31 experienced, 12 about to receive Master's degree
- Chinese Teachers: 5 schools chosen; all teachers interviewed at these schools
- Topics for interview: Subtraction with Regrouping, Multi-digit Multiplication, Division by Multiples, and Comparison between Area and Perimeter

Behind the Scenes

1. Many Chinese elementary teachers are math specialists
2. Teachers are rotated through all grade levels approximately every 4 years
3. Many Chinese elementary teachers in Ma's study teach 4-5 45-minute lessons a day
4. The rest of the time is spent planning, observing, and deliberately improving mathematics teaching

Foreword #1

"This book appears to be a comparative study of American and Chinese teachers of mathematics, but its most important contributions are not comparative, but theoretical."

Foreword #2

"This book appears to be about the practice of mathematics teaching, but it demands a hearing among those who set policy for teaching and teacher education."
### Foreword

- "This book appears to be most relevant to the pre-service preparation of teachers but its most powerful findings may well relate to our understanding of teachers' work and their career-long professional development."

### Pre-assessment

- Please take 20-25 minutes to complete the pre-assessment based on scenarios we will read about in the book.
- Please write your initials on the top of the page.

### Subtraction

Please solve the following problem, keeping in mind the strategies that you use.

\[
63 - 29 =
\]

### Subtraction

Now consider the problem:

\[
15 - 7 =
\]

Are the strategies discussed in the previous problem applicable to this problem?
Guiding Questions for Chapter #1

For the next session, please read Chapter 1, "Subtraction with Regrouping: Approaches to Teaching A Topic". Consider the following guiding questions while reading:

1. What content is necessary for students to understand before learning subtraction with regrouping?

2. On page 2, Ma asks the question, "Would a teacher’s subject matter knowledge make any difference in his or her teaching, and eventually contribute to students’ learning?" After reading chapter 1, how would you respond to Ma?

3. Is Ma implying that we should not teach procedurally?

4. What instructional methods seemed to be most effective? Why?
## Facilitating Guide Session #2

### Activity/Discussion

<table>
<thead>
<tr>
<th>Introduction</th>
<th>Time Frame</th>
</tr>
</thead>
</table>
| - Pass around sign-in sheet, snacks  
- Teachers get into assigned groups  
- Reintroduce each other in small groups | ~5-10 min |

<table>
<thead>
<tr>
<th>Group Discussion:</th>
<th>Time Frame</th>
</tr>
</thead>
</table>
| - What is your overall impression of the book so far?  
- Did the reading leave you with any questions so far?  
- What are the “deep understandings of mathematics” that students “should” take away from subtraction with regrouping? | ~15 min |

(Power Point slide 2-3)  
(Recording Sheet)

<table>
<thead>
<tr>
<th>Large group share-out, I record</th>
<th>Time Frame</th>
</tr>
</thead>
<tbody>
<tr>
<td>~10 min</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Subtraction with regrouping: Methods</th>
<th>Time Frame</th>
</tr>
</thead>
</table>
| - In large group, go through methods 1-6  
(Power Point slides 4-9)  
- In small groups, discuss methods  
  - Are we omitting any?  
  - Do all methods promote the same understanding?  
  - Does each method support students in developing all of the deep understandings discussed earlier?  
  - Which methods would you use? | ~25 min |

(Power Point slides 10-11)  
(Methods Sheet)

<table>
<thead>
<tr>
<th>Ma’s Knowledge Package</th>
<th>Time Frame</th>
</tr>
</thead>
</table>
| - Discuss three different levels  
- Importance of Level 1 (borrowing won’t work, which methods will work?) | ~20 min |

(Power Point slide 12)

Give assignment for next week: Guiding Questions Chapter 2  
~3 min

Attachment 1: Power Point Presentation 2  
Attachment 2: Recording Sheet  
Attachment 3: Methods Sheet  
Attachment 4: Guiding Questions Chapter 2
Deepening Fundamental Mathematics

Group Discussion

- Choose a recorder
- 1. Take turns describing your overall impression of the book so far
- 2. Did the reading leave you with any questions? Concerns?

Group discussion

- What are the "deep understandings of mathematics" that students "should" take away from subtraction with regrouping?
- Discuss general ideas more in depth. Where should students be in their development of certain concepts?

Solving Subtraction with Regrouping: Method 1

- [Image of subtraction problem-solving process]

Solving Subtraction with Regrouping: Method 2

- [Image of subtraction problem-solving process]
Subtraction with Regrouping

- Do all methods promote the same “deep understandings of mathematics”? Explain.

- Which of the methods would you use to teach subtraction with regrouping?

Mac's Knowledge Package

- 1st Level: minuends between 10 and 20
- 2nd Level: minuends between 19 and 100
- 3rd Level: minuends larger than 99

Solving Subtraction with Regrouping: Method A

\[
\begin{array}{c}
29 \\
+ 7 \\
\hline
36 \\
+ 1 \quad \text{add 1 to 34} \\
\hline
37
\end{array}
\]

\[
\begin{array}{c}
63 \\
- 20 \\
\hline
43 \\
- 9 \\
\hline
34
\end{array}
\]

Are we forgetting any methods?
1. Describe your overall impression of the book so far.

2. Did the reading leave you with any questions? If so, what are they?

3. What are the "deep understandings of mathematics" that students "should" take away from Subtraction-With-Regrouping?
Methods of Subtraction with Regrouping
Session 2

1. Decompose 63 to 5 groups of 10 and 3 ones — then subtracted each digit of the subtrahend.

\[
\begin{array}{c}
63 \\
-29 \\
\hline
34
\end{array}
\]

2. Shifted both values up one to simplify the problem: 64 - 30.

\[
\begin{array}{c}
63 \\
-30 \\
\hline
33
\end{array}
\]

Mentally change the subtrahend to 30, which makes subtraction easier but remember that one less due to the result.

3. 

\[
\begin{array}{c}
60 \\
-30 \\
\hline
30
\end{array}
\]

\[
\begin{array}{c}
+1 \\
\hline
31
\end{array}
\]

\[
\begin{array}{c}
+3 \\
\hline
34
\end{array}
\]

4. 

\[
\begin{array}{c}
29 \\
+1 \\
\hline
30
\end{array}
\]

\[
\begin{array}{c}
+10 \\
+10 \\
\hline
40
\end{array}
\]

\[
\begin{array}{c}
+10 \\
+50 \\
\hline
60
\end{array}
\]

\[
\begin{array}{c}
+3 \\
\hline
63
\end{array}
\]

\[
\begin{array}{c}
+33 \\
\hline
63
\end{array}
\]

\[
33 + 1 = 34
\]
Guiding Questions for Chapter #2

For the next session, please read Chapter 2, "Multidigit Number Multiplication: Dealing With Students' Mistakes". Consider the following guiding questions while reading:

1. How important is it that 0 be used as a place-holder?

2. What is the role of 0 in mathematics? What understanding(s) should 6th-grade students have of 0?

3. Do you consider a “knowledge package” when teaching Multi-digit Multiplication? What would this “knowledge package” consist of?
## Facilitating Guide Session #3

<table>
<thead>
<tr>
<th>Activity/Discussion</th>
<th>Time Frame</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Introduction</strong></td>
<td>~3-5 min</td>
</tr>
<tr>
<td>• Review impressions and questions from last session</td>
<td></td>
</tr>
<tr>
<td>(Power Point slide 2-3)</td>
<td></td>
</tr>
<tr>
<td><strong>Group Discussion: Chapter 2</strong></td>
<td>~5 min</td>
</tr>
<tr>
<td>• What surprised you? What concerned you?</td>
<td></td>
</tr>
<tr>
<td>(Power Point slide 4)</td>
<td></td>
</tr>
<tr>
<td><strong>Large group share-out, I record</strong></td>
<td>~5 min</td>
</tr>
<tr>
<td><strong>Instructional Methods Discussion</strong></td>
<td>~7 min</td>
</tr>
<tr>
<td>• What instructional method for dealing with the student misconception most appealed to you? Why?</td>
<td></td>
</tr>
<tr>
<td>(Power Point slide 5)</td>
<td></td>
</tr>
<tr>
<td><strong>Zero: Discussion/Share-out</strong></td>
<td>~5 min</td>
</tr>
<tr>
<td>• How important is it that zero be used as a placeholder?</td>
<td></td>
</tr>
<tr>
<td>(Power Point slide 6)</td>
<td></td>
</tr>
<tr>
<td><strong>The Concept of Zero: Large Group Discussion</strong></td>
<td>~10 min</td>
</tr>
<tr>
<td>• Where are students in their conceptual understanding of zero in 2&lt;sup&gt;nd&lt;/sup&gt; grade? 6&lt;sup&gt;th&lt;/sup&gt; grade? 12&lt;sup&gt;th&lt;/sup&gt; grade?</td>
<td></td>
</tr>
<tr>
<td>• Do students know where a particular “treatment” of zero falls into the “big picture” of mathematics?</td>
<td></td>
</tr>
<tr>
<td>• What do we expect students to understand about zero?</td>
<td></td>
</tr>
<tr>
<td>(Power Point slides 7-8)</td>
<td></td>
</tr>
<tr>
<td><strong>The Concept of Zero: Small Group Discussion</strong></td>
<td></td>
</tr>
<tr>
<td>1. Brainstorm the meaning(s) of zero.</td>
<td>1. ~5 min</td>
</tr>
<tr>
<td>2. Where do we see zero in elementary and secondary mathematics?</td>
<td>2. ~5 min</td>
</tr>
<tr>
<td>3. What are common misconceptions about zero?</td>
<td>3. ~5 min</td>
</tr>
<tr>
<td>4. Create a graphic organizer titled “The Meaning of Zero”</td>
<td>4. ~15 min</td>
</tr>
<tr>
<td>(Power Point slides 9-11)</td>
<td></td>
</tr>
<tr>
<td><strong>Dealing with Student Misconceptions: Small group discussion</strong></td>
<td>~5-10 min</td>
</tr>
<tr>
<td>A student was asked the question:</td>
<td></td>
</tr>
<tr>
<td>“How does the number 5.20 compare with the number 5.2?”</td>
<td></td>
</tr>
<tr>
<td>The student responded:</td>
<td></td>
</tr>
<tr>
<td>“5.20 is larger because 20 is bigger than 2.”</td>
<td></td>
</tr>
<tr>
<td>What is the student’s misconception and how would you go about correcting it?</td>
<td></td>
</tr>
<tr>
<td>(Power Point slide 12)</td>
<td></td>
</tr>
<tr>
<td><strong>Give assignment for next week: Guiding Questions Chapter 3</strong></td>
<td>~3 min</td>
</tr>
</tbody>
</table>

Attachment 1: Power Point Presentation 3  
Attachment 2: Guiding Questions Chapter 3
Deepening Fundamental Mathematics

A BOOK STUDY

Reactions to Chapter 1: Overall Impression
- Redundant
- Interesting: Multiple ways of (subtraction with) regrouping
- Slanted
- Vocabulary
- 10-year-old data
- Comparisons
- Subtraction within 20

Reactions to Chapter 1: Questions
- In other countries, how long is a school day/year?
- Is there a more current follow-up study?
- What is involved in a knowledge package? (Vague)
- Where is the appropriate placement for the "use of algorithm"?
- What is the reasoning for our instruction?
- Manipulatives?

Group Discussion on Chapter 2
- Choose a recorder
- Discuss Chapter 2: Multi-digit Multiplication
  - What surprised you?
  - What concerned you?
  - Are you left with any questions?

Instructional Methods
1. What instructional method for dealing with the given misconception most appealed to you and why?
2. What constitutes teaching in a conceptual way versus a procedural way?

Group Discussion
How important is it that zero be used as a placeholder?

Pages 17-28: Discuss the different ways teachers in the study handled the student misconception.
What about ZERO?

The Concept of Zero

- Where are students in their conceptual understanding of zero when they leave 2nd grade? 6th grade? 12th grade?
- Do students know where a particular "treatment" of zero falls into the big picture of mathematics?
- What do we expect students to understand about zero?

The Concept of Zero

1. Brainstorm with your group the meaning(s) of zero.

2. Where do we see zero in elementary and secondary mathematics?

3. Where are common misconceptions or uncertainties about zero?

The Concept of Zero

Let's put some organization to the abstract concept of "the meaning of zero?"

4. Create a graphic organizer/flowchart/knowledge package titled "The Meaning of Zero"

(For more of this activity should promote student understanding)

Dealing with a Student Misconception

A student was asked the question:
"How does the number 5.20 compare with the number 5.2?"

The student responded:
"5.20 is larger because 20 is bigger than 2."

What is the student's misconception and how would you go about correcting it?
Guiding Questions for Chapter #3

For the next session, please read Chapter 3, "Generating Representations: Division by Fractions". Consider the following guiding questions while reading:

1. How would you explain the rationale behind the algorithm “multiply by the reciprocal”?

2. What are the different understandings of fractions that students need to understand in order to understand division by fractions?

3. How can division of fractions be linked to previous concepts?
## Facilitating Guide Session #4

### Activity/Discussion

<table>
<thead>
<tr>
<th>Introduction</th>
<th>Time Frame</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Review impressions and questions from last session</td>
<td>~3-5 min</td>
</tr>
<tr>
<td>• Review Meanings of Zeros from last session</td>
<td></td>
</tr>
<tr>
<td>(Power Point slide 2-3)</td>
<td></td>
</tr>
<tr>
<td>Group Discussion: Chapter 3</td>
<td>~10 min</td>
</tr>
<tr>
<td>• What surprised you? What concerned you?</td>
<td></td>
</tr>
<tr>
<td>(Power Point slide 4)</td>
<td></td>
</tr>
<tr>
<td>Algorithms for Dividing by Fractions</td>
<td>~15 min</td>
</tr>
<tr>
<td>• In small groups, justify each step in each method</td>
<td></td>
</tr>
<tr>
<td>• How would you describe the rationale behind “multiply by the reciprocal” to a 6th grade student?</td>
<td></td>
</tr>
<tr>
<td>(Power Point slide 5-8)</td>
<td></td>
</tr>
<tr>
<td>(Making Sense of the Algorithm Sheet)</td>
<td></td>
</tr>
<tr>
<td>Models of Division by Fractions</td>
<td>~25 min</td>
</tr>
<tr>
<td>• Measurement Model</td>
<td></td>
</tr>
<tr>
<td>• Partitive Model</td>
<td></td>
</tr>
<tr>
<td>• Factors and Products Model</td>
<td></td>
</tr>
<tr>
<td>• Case Analyses: Have the students correctly represented the fractional division problem?</td>
<td></td>
</tr>
<tr>
<td>• Group Work: In small groups create 3 word problems representing the same division by fractions problem. Each problem should represent a different model.</td>
<td></td>
</tr>
<tr>
<td>(Power Point slides 9-13)</td>
<td></td>
</tr>
<tr>
<td>(Different Models of Division by Fractions Sheet)</td>
<td></td>
</tr>
<tr>
<td>Large Group Discussion</td>
<td>~5-10 min</td>
</tr>
<tr>
<td>• How does fractional division relate to previous concepts?</td>
<td></td>
</tr>
<tr>
<td>(Power Point slide 14)</td>
<td></td>
</tr>
<tr>
<td>Case Study</td>
<td>~15 min</td>
</tr>
<tr>
<td>• Play video clip of student demonstrating fractional division (Hand out typed dialog)</td>
<td></td>
</tr>
<tr>
<td>• Discussion: What is the nature of the student's understanding of fractional division?</td>
<td></td>
</tr>
<tr>
<td>o Prompt discussion of partial understandings</td>
<td></td>
</tr>
<tr>
<td>(Power Point slide 15)</td>
<td></td>
</tr>
<tr>
<td>Give assignment for next week: Guiding Questions Chapter 4</td>
<td>~3 min</td>
</tr>
</tbody>
</table>

### Time Frame
- 3-5 min
- 10 min
- 15 min
- 25 min
- 5-10 min
- 15 min
- 3 min

**Attachment 1:** Power Point Presentation 4  
**Attachment 2:** Making Sense of the Algorithm Sheet  
**Attachment 3:** Different Models of Division by Fractions Sheet  
**Attachment 4:** Video Clip Sheet  
**Attachment 5:** Guiding Questions Chapter 4
Power Point Presentation 4

Reactions to Chapter 2: Multi-Digit Multiplication
- More interesting to read
- Relevant
- Repetitive (Conceptual vs. Procedural)
- Negative/Insulting
- Scary/Shocking
- Debater: Being picky with words
- good practice vs. not even look at problem
- Distributive Property

"What is/are the meaning(s) of Zero"?

Group 1
- When using any equation: z = 1
- Equals or equal: 2 = 2 + 0
- Group 2
- Zero has meaning
- Parent function: x
- Showing combinations
- In order to solve
- Learning problems on assigned
- Using variables
- Getting the answer
- Zero in a coordinate plane
- Writing the equation of a line

Reactions to Chapter 3
- What surprised you?
- What interested you?
- What concerns you?
- What questions do you have?

Algorithms for Dividing by Fractions

Invert and Multiply
In your group, record justification for each step

Method 1
1/2 ÷ 6 = (1/2) ÷ (6) = (1/2) ÷ 1
= (1/2) x 6
= 3/2

Method 2
1/2 ÷ 1/2 = (1/2) ÷ (1/2) = (1/2) ÷ 1
= (1/2) x 2
= 1/2

Using the Distributive Property

In your group, record justification for each step

Method 3
1/2 ÷ 1/2 = (1/2) ÷ (1/2) = (1/2) ÷ 1
= (1/2) x 2
= 1/2

Method 4
1/2 ÷ 1/2 = (1/2) ÷ (1/2) = (1/2) ÷ 1
= (1/2) x 2
= 1/2

- 106
Algorithms for Dividing by Fractions

Alternatives to Multiplying by Reciprocal

In your group, record justification for each step:

\[
\begin{align*}
&\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} + \frac{1}{4} \\
&\text{Marked A} \\
&\frac{1}{2} + \frac{1}{4} = \frac{1}{2} + \frac{1}{2} \\
&= \frac{1}{2} \\
&= \frac{3}{4}
\end{align*}
\]

\[
\begin{align*}
&\frac{1}{4} \div \frac{1}{2} = \frac{1}{4} + \frac{1}{2} \\
&\text{Marked B} \\
&\frac{1}{4} + \frac{1}{2} = \frac{1}{4} + \frac{1}{2} \\
&= \frac{1}{4} + \frac{1}{2} \\
&= \frac{1}{4} + \frac{2}{4} \\
&= \frac{3}{4}
\end{align*}
\]

Discussion

- How would you explain the rationale behind "multiply by the reciprocal?"
  - In what ways given in the book (if any) make sense most to you?
  - In what ways would you explain the algorithm to a 6th grade student?

Models of Division by Fractions

The Measurement Model

Finding how many \( \frac{1}{2} \)'s there are in \( 1 \frac{1}{2} \) or finding how many times \( 1 \frac{1}{2} \) is of \( \frac{1}{2} \):

- \( 1 \frac{1}{2} \text{ feet} \div \frac{1}{2} \text{ feet} = \frac{1}{2} \)

- "How many \( \frac{1}{2} \)-foot lengths are there in something that is 1 \( \frac{1}{2} \) foot long?"

Models of Division by Fractions

The Partitive Model

Finding a number such that \( \frac{1}{2} \) of it is \( 1 \frac{1}{2} \):

- \( 1 \frac{1}{2} \text{ feet} \div \frac{1}{2} \text{ feet} = \frac{1}{2} \text{ feet} \)

- "If half a length is \( 1 \frac{1}{2} \) feet, how long is the whole?"

Models of Division by Fractions

Factors and Product

Finding a factor that multiplies by \( \frac{1}{2} \) will make \( 1 \frac{1}{2} \):

- \( 1 \frac{1}{2} \text{ square feet} \div \frac{1}{2} \text{ feet} = 3 \text{ feet} \)

- "If one side of a \( 1 \frac{1}{2} \) square foot rectangle is \( \frac{1}{2} \) feet, how long is the other side?"

Models of Division by Fractions

- Case analyses:
  - Read through sample students responses to the following tasks.
  - Determine whether the student has correctly represented the problem.
  - If they have, identify the model they are using.
Group Work
In groups, create 3 word problems representing the same division by fraction problem. Each word problem should represent a different model of division by fractions.

Discussion
How does division of/by fractions relate to previous concepts?
- What connections should students make?
- What knowledge do students build off of?
- What problems would you assign to students before assigning a division by fraction problem?

Case Study
Video Clip Dividing by Fractions
- Describe the nature of the students' understanding of fraction division.
  Students can have a sense of difficult concepts and partial understandings can exist at the same time.
Making Sense of the Algorithm Sheet
Session 4

"Invert and Multiply":

Method 1

\[ 1\frac{3}{4} + \frac{1}{2} = 1\frac{3}{4} + (1 + 2) \]
\[ = 1\frac{3}{4} + 1 \times 2 \]
\[ = 1\frac{3}{4} \times 2 + 1 \]
\[ = 1\frac{3}{4} \times (2 + 1) \]
\[ = 1\frac{3}{4} \times 2 \]

Method 2

\[ 1\frac{3}{4} + \frac{1}{2} = (1\frac{3}{4} \times \frac{2}{1}) + (\frac{1}{2} \times \frac{2}{1}) \]
\[ = (1\frac{3}{4} \times \frac{2}{1}) + 1 \]
\[ = 1\frac{3}{4} \times \frac{2}{1} \]
\[ = 3\frac{1}{2} \]
Using the Distributive Property:

Method 3

\[ 1 \frac{3}{4} + \frac{1}{2} = (1 + \frac{3}{4}) + \frac{1}{2} \]
\[ = (1 + \frac{3}{4}) \times \frac{2}{1} \]
\[ = (1 \times 2) + (\frac{3}{4} \times 2) \]
\[ = 2 + 1\frac{1}{2} \]
\[ = 3\frac{1}{2} \]

Method 4

\[ 1 \frac{3}{4} + \frac{1}{2} = (1 + \frac{3}{4}) + \frac{1}{2} \]
\[ = (1 + \frac{1}{2}) + (\frac{3}{4} + \frac{1}{2}) \]
\[ = 2 + 1\frac{1}{2} \]
\[ = 3\frac{1}{2} \]
Alternatives to Multiplying by Reciprocal:

Method 5

\[
1\frac{3}{4} + \frac{1}{2} = \frac{7}{4} + \frac{1}{2}
\]
\[
= \frac{7 + 1}{4 + 2}
\]
\[
= \frac{7}{2}
\]
\[
= 3\frac{1}{2}
\]

Method 6

\[
1\frac{3}{4} + \frac{1}{2} = \frac{7}{4} + \frac{1}{2}
\]
\[
= (7 \div 4) + (1 \div 2)
\]
\[
= 7 + 4 + 1 \times 2
\]
\[
= 7 + 1 + 4 \times 2
\]
\[
= (7 + 1) + (4 + 2)
\]
\[
= \frac{7 + 1}{4 + 2}
\]
The Measurement Model:
Finding how many $\frac{1}{2}$'s there are in $1\frac{3}{4}$ or finding how many times $1\frac{3}{4}$ is of $\frac{1}{2}$.

$1\frac{3}{4}$ feet $+ \frac{1}{2}$ feet $= \frac{3}{2}$

"How many $\frac{1}{2}$-foot lengths are there in something that is 1 and $\frac{3}{4}$ feet long?"

The Partitive Model of Division:
Finding a Number such that $\frac{1}{2}$ of it is $1\frac{3}{4}$

$1\frac{3}{4}$ feet $+ \frac{1}{2}$ = $\frac{3}{2}$ feet

"If half a length is 1 and $\frac{3}{4}$ feet, how long is the whole?"

Factors and Product:
Finding a factor that multiplied by $\frac{1}{2}$ will make $1\frac{3}{4}$

$1\frac{3}{4}$ square feet $+ \frac{1}{2}$ feet = $\frac{3}{2}$ feet

"If one side of a $1\frac{3}{4}$ square foot rectangle is $\frac{1}{2}$ feet, how long is the other side?"
Students A, B, and C were asked to create word problems to describe the division by fraction problems $\frac{5}{7} \div \frac{2}{7}$ and $\frac{3}{4} \div \frac{1}{8}$. Read their responses and determine if they have come up with a correct word problem, and if so, what model they are using.

\[
\frac{5}{7} \div \frac{2}{7}
\]

Student A: 
"At the party there were 7 children and 7 slices of pizza, but everyone wasn’t hungry. Now there are 5 pieces left and 2 kids want to split them. How much does each kid get?"

Student B: 
"Jane has $\frac{5}{7}$ of a pound of grapes from the story and wants to wash and store them in separate Ziploc bags. However, her Ziploc bags can only hold $\frac{2}{7}$ pounds each. How many bags would Jane divide her grapes into if she fills one bag as much as possible before moving onto the next?"

Student C: 
"Every year Mrs. Alice gets pencils for her 7 students. The pencils she bought last year came in a pack of 5, but this year they only come in packs of 2. How many extra pencils will she have to buy this year?"

\[
\frac{3}{4} \div \frac{1}{8}
\]

Student A: 
"At a birthday party the mother is dividing the cake between the 8 kids. Only $\frac{3}{4}$ of the cake is left but only 1 wants it now. How much cake will the 1 kid get?"

Student B: 
"A graduate school teacher is writing a very difficult $\frac{3}{4}$ page paper on teaching elementary school children math. She is struggling with the paper, and can only write $\frac{1}{8}$ of a page per hour. How many hours will it take for her to write the entire paper?"

Student C: 
"Joe’s car’s gas tank is broken down into $\frac{1}{4}$th’s of a tank. His car’s tank is $\frac{3}{4}$th full. His friend doesn’t understand how much $\frac{3}{4}$ of a tank is because her car’s tank is broken up into $\frac{1}{8}$ths. She wants to compare how much gas is in Joe’s car by using her tank’s $\frac{1}{8}$ths. How many $\frac{1}{8}$ths of gas does Joe have?"
Problem #1: \[ 1 \div \frac{1}{3} \]

Teacher: Can you work that problem Elliot?

(Elliot quickly writes 3)

Teacher: And how did you get that so fast?

Elliot: Um, one third goes into one three times because there’s three pieces in one whole

(Elliot draws a rectangle and shades one third of it. He draws another rectangle split into thirds)

Elliot: I cut that into thirds three times because I’ve got one of these, this will connect to that one, this will connect to that one, and this will connect to that one.

(Elliot draws lines showing one section (one third in the first rectangle) going to one section (one third in the second rectangle))

Teacher: Is that what division is?

Elliot: Yeah how many times that \((\frac{1}{3})\) goes into that \((1)\).

Problem #2: \[ 1 \frac{1}{2} + \frac{1}{3} \]

(Elliot writes down 4, then thinks, then writes \(\frac{1}{6}\), giving an answer of \(4 \frac{1}{6}\))

Teacher: Can you explain your answer?

Elliot: Well all you did was add one half, so the answer was 3 if I did not have, if this was not there (circling the \(\frac{1}{2}\)) the answer would be three, but that is there, and one third goes into one half one time and now then I’ve got one sixth left, two sixths equals one third, and three sixths equals one half so I take away two sixths because I’m taking away a third out of the one half and I have one sixth left.

Teacher: I see just how you thought about that.
Guiding Questions for Chapter #4
For the next session, please read Chapter 4, "Exploring New Knowledge: The Relationship Between Perimeter and Area ". Consider the following guiding questions while reading:

1. How important is confidence in succeeding at math?
2. Where do elementary students see “proof” in mathematics?
3. How can a teacher promote student mathematical inquiry beyond that of his/her own knowledge?
4. How important is the shaping of students’ thinking? Should we encourage students to think in an organized fashion?
## Facilitating Guide Session #5

### Activity/Discussion

<table>
<thead>
<tr>
<th>Introduction</th>
<th>3-5 min</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Any new impressions or questions? (Power Point slide 2)</td>
<td></td>
</tr>
<tr>
<td><strong>Large Group Discussion</strong></td>
<td>7 min</td>
</tr>
<tr>
<td>- How important is confidence in succeeding at math? (Power Point slide 3)</td>
<td></td>
</tr>
<tr>
<td><strong>Small Group Discussion (then share out)</strong></td>
<td>15 min</td>
</tr>
<tr>
<td>- How important is the shaping of student thinking?</td>
<td></td>
</tr>
<tr>
<td>- Should we encourage students to think in an organized fashion?</td>
<td></td>
</tr>
<tr>
<td>- What “habits of mind” should students take away from math?</td>
<td></td>
</tr>
<tr>
<td>- What do teachers do to promote “habits of mind”? (Power Point slides 4-5)</td>
<td></td>
</tr>
</tbody>
</table>

### Presentation on Chapter 5: Profound Understanding of Fundamental Mathematics

- Connectedness
- Multiple Perspectives
- Basic Ideas
- Longitudinal Coherence
- Examples of assumptions by colleges in the U.S.
  - Elementary teachers are given breadth, not depth of knowledge
  - Elementary teachers should already know the basics when entering college
- Discussion: Would reform in teacher education be beneficial? Possible? (Power Point slides 6-13)

### Presentation on Chapter 6

- When is PUFM attained? Throughout teaching career when given time to study materials!
- Circumstances of U.S. versus Chinese teachers (Power Point slides 14-16)

### Presentation on Chapter 7

- Value of text book and curriculum (Power Point slide 17-18)

### Post-Assessment and Survey (Power Point slide 19)

~20-25 min

---

Attachment 1: Power Point Presentation 5
Appendix A: Post-Assessment
Appendix B: Workshop Evaluation Survey
How important is confidence in succeeding at math?
In what ways does lack of confidence hurt students (and/or teachers)?

Group Discussion: Please choose a recorder
• How important is the shaping of student thinking?
• Should we encourage students to think in an organized fashion?
• What "habits of mind" should students take away from math?

Group Discussion: Please choose a recorder
• What do teachers do to promote these "habits of mind"? Give specific examples. (i.e. Instructional methods; assignments)
• Do students realize that they are being taught these "habits of mind"? Should they?

Chapter 5: Profound Understanding of Fundamental Mathematics (PUFM)
1. Connectedness
• Students will learn a unified body of knowledge
Chapter 5: Profound Understanding of Fundamental Mathematics (PUFM)

2. Multiple Perspectives
- Appreciate different facets of an idea and various approaches
- Know advantages and disadvantages
- Leads to a flexible understanding

3. Basic Ideas
- Awareness of "simple but powerful" basic concepts and principles of mathematics
- Review/Reinforce those basic ideas

4. Longitudinal Coherence
- Not limited to knowledge in a certain grade, but whole elementary mathematics curriculum
- Lay proper foundation for future concepts

Summary
- Breadth, Depth, and Thoroughness are necessary
- "Elementary Mathematics is constructed very differently in China and in the U.S."

Is it an assumption in the U.S. that elementary mathematics is "basic" and "commonly-understood"?
Chapter 5: Profound Understanding of Fundamental Mathematics (PUFM)

Forward to book: "This book appears to be about the practice of mathematics teaching, but it seems a teaching guide. Is it possible?"

Chapter 6: When and How is it Attained?

- "Studying teaching materials intensively when teaching it"
- Only teaching, or only studying is not as effective (paraphrased)
- Teaching, teaching "round-by-round", studying teaching materials

Circumstances

<table>
<thead>
<tr>
<th>Chinese Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 teaching</td>
</tr>
<tr>
<td>4 hours a day</td>
</tr>
<tr>
<td>4.5 hours a day to study math</td>
</tr>
<tr>
<td>20 weeks, a semester, 42 weeks, 240 hours to study math</td>
</tr>
<tr>
<td>Teaching subjects at school</td>
</tr>
<tr>
<td>0.5 hours daily to study math</td>
</tr>
</tbody>
</table>

a = number of hours teaching

S = 0 + 4.5x

S = 240 + 0.5x

After 60 days of teaching, elementary teachers in China have expected to study math.

Chapter 7: Conclusion

Understand the role that curriculum materials, including textbooks, might play in reform

"educators often disparage textbooks, and many reform-oriented teachers repudiate them, announcing disdainfully that they do not use texts. This idealization of professional autonomy leads to the view that good teachers do not follow textbooks, but instead make their own curriculum... This hostility to texts, and the idealized image of the individual professional, have inhibited careful consideration of the constructive role that curriculum might play.”
Post-assessment and survey

Please write your initials on the top of the post-assessment.
Please take time to complete the post-assessment involving math problems we have discussed over the past 5 sessions.
Please also take time to fill out the end-of-workshop survey.

Thank you so much for attending and have a great evening!
Appendix E
Student Work

Response 1

I have 1 3/4 lbs of candy and I want to give each person at the party 1/16 lbs. How many people can I serve? How much candy is left over?

3 people
1/16 left over

Response 2

You have one full snack bar and 3 1/4 of another. How many pieces will you have if you cut lengths equal to half of the full snack bar?

1 3/4
4/4 ÷ 2 = 7 2 = 7/2 = 3.5

3 groups of 3/4
8 of 1/4 (1/4 is 1/2 of 1/2) so 3.5

Response 3

I would ask her if this works for all closed figures. I would ask her to try convex x concave figures with the same perimeter (use graph paper)

Ex: no

Also, think of other factor pairs of 32. Should a 1 x 32 rectangle have a perimeter of 24?
Sample Response Pre-Assessment

1. How would you explain to a group of second graders how to solve the following problem?

```
  52
+ 27
---
  79
```

We must look at the ones column first. Since seven is greater than two, we must borrow a group of ten from the tens column. We will make the two into 12. Then, subtract 7 from 12. Write five in the ones column. When we borrow one group of ten, we have four groups of ten left. Subtract two from four and there are two left.

This down. The answer is 79.

2. Explain the mathematical concepts you would review to help correct the mistake in the following problem:

```
  123
x 645
---
  73800
  615
  4920
  73800
---
  640065
```

The zeros in the second and third row are missing. When you move to the 4, you are multiplying 723 by 640. You would put zero as a placeholder and multiply the 4 by 3, then the 4 by 2, and then the 4 by 1 (2nd row-4920). We are moving the the hundred place (123x600). There will be two zeros in our answer so we need to put them in as placeholders and multiply 6 by 3, then 6 by 2, then 6 by 1.

This will equal 73800. Then we add all three numbers together.
3. Write a word problem to represent $1 \frac{1}{4} + \frac{1}{2}$. You need to devise a problem and solve the problem you wrote.

$$\frac{7}{4} \cdot \frac{2}{1} = \frac{14}{4} = \frac{3}{2} + \frac{3}{2}$$

Tell you multiply by $\frac{1}{2}$, it's the same as dividing by 2.

If you need more time, divide by $\frac{1}{2}$, it's the same as multiplying by 2.

I'm Struggling!

4. Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you the following picture to prove what she is doing. How would you respond to this student?

I would respond that she is correct. The longer the length or width becomes, the area increases as well. (I might use the idea as a fence for perimeter and a yard as a representation for area.)

Liping Ma (1999) Knowing and Teaching Elementary Mathematics
Sample Response Post-Assessment

1. How would you explain to a group of second graders how to solve the following problem?

\[
\begin{array}{c}
32 \\
-27 \\
\hline
5 \\
\end{array}
\]

I would add three to 27 to make it 30. Then, I would add the same amount to 32 (3) to get 35.

The answer would be 35.

These numbers would be easier to calculate.

2. Explain the mathematical concepts you would review to help correct the mistake in the following problem:

\[
\begin{array}{c}
123 \\
\times 645 \\
615 \\
\times 645 \\
492 \\
\times 645 \\
738 \\
\times 645 \\
1845 \\
\hline
615 \\
+4500 \\
+12900 \\
+1935 \\
\hline
79335
\end{array}
\]

I could do this problem as an array which better emphasizes place value.
3. Write a word problem to represent $1 \frac{1}{4} \div \frac{1}{2}$. You need to devise a problem and solve the problem you wrote.

$$1 \frac{1}{4} \div \frac{1}{2} = \frac{3}{4} \div \frac{1}{2}$$

4. Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you the following picture to prove what she is doing. How would you respond to this student?
Sample Response Evaluation Survey

Workshop Evaluation Survey

Please answer the following questions on a scale of one to four, with one being “The workshop did not meet its goal”, and four being “The workshop met its goal very well”. Then, please explain your choice in terms of your own experience. Thank you!

1. How well did this workshop meet its goal of deepening your understanding of subtraction with regrouping, multi-digit multiplication, division by fractions, and comparing area and perimeter?

   1  2  3  4

   Please Explain:
   
   It definitely allowed me to see other ways to solve problems and gain new perspectives.

2. How well did this workshop meet its goal of improving your mathematical instructional methods?

   1  2  3  4

   Please Explain:
   
   I didn't.

3. How well did this workshop meet its goal of setting into motion a dedication to the deliberate study of the mathematics we teach?

   1  2  3  4

   Please Explain:
   
   It's amazing to see how many methods that we take for granted and allowed us to analyze them more thoroughly.
Please answer the following question on a scale of one to four with one being “not relevant” and four being “very relevant”.

4. How relevant was the reading to your everyday practice?
   1  2  3  4
   Please Explain:
   It opened my eyes to common (and adults') misconceptions and ways to improve on them.

5. How relevant were the discussions to your everyday practice?
   1  2  3  4
   Please Explain:
   I will definitely include some of these deep understandings in my discussions and encourage students to think of different ways of thinking, not just one set way.