A Mathematics Curriculum Study In Trigonometric Graphs To Align With The Common Core Standards

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A Mathematics Curriculum Study In Trigonometric Graphs To Align
With The Common Core Standards

by

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A thesis submitted to the Department of Education and Human Development of the State
University of New York College at Brockport
in partial fulfillment of the requirements for the degree of
Master of Science in Education
Abstract

The Common Core State Standards (CCSS) were written by The National Governors Association (NGA), in collaboration with the Council of Chief State School Officers (CCSSO) in 2010, and was adopted by 45 states in an attempt to nationalize the United States curriculum in English and mathematics. Now it is up to the states to implement new curriculum to align to these standards.

In the third year of high school mathematics, students are introduced to radian measures and trigonometric functions, and these abstract mathematical topics can be difficult for them to learn.

The lesson plans contained within this curriculum project are aligned to current CCSS related to trigonometric functions and modeling and are intended for teachers to use in their classrooms to visualize and model the basic trigonometric functions beginning with an introduction to radians, and including the functions sine cosine and tangent.
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Chapter 1: Introduction

The Common Core State Standards (CCSS) are due for full implemented in most high schools across the United States by the 2014-2015 school year (Rust, 2012). According to EngageNY (2013), this means that New York State will begin to implement the new state Regents Exams in mathematics aligned to Common Core as early as the 2013-2014 school year. Common Core testing began in New York State middle schools in the 2012-2013 school year (EngageNY, 2013) and other states are expected to do likewise in the coming years (Rust, 2012). This change may be difficult for some teachers to adapt to because in many instances, teachers may be required to restructure entire units of study to align with the new standards.

Significance of the Problem

This curriculum project is intended to provide a useable 12 day unit to introduce high school students to the unit circle, radian measure, graphs, and applications of the three basic trigonometric functions that aligns with the new standards. It is designed with the purpose of making an easier shift to the new standards, and to serve as a model for matching curriculum to standards.

In 2010, the National Governors Association (NGA), in collaboration with the Council of Chief State School Officers (CCSSO) published the Common Core State Standards (National Governors Association Center for Best Practices, 2010). According to Achieve, Inc., a high school diploma has lost its credibility over the years because the material required for earning a diploma has no connection to the knowledge and skills required in a student’s post-high school life (Achieve, 2004). One key goal stated in the CCSS is to make sure all students are prepared to compete in our global economy. The NGA and CCSSO compared the United States to other top performing countries on international tests such as the Trends in International Mathematics and Science Study (TIMMS) as evidence that national standards should be developed (National
Governors Association Center for Best Practices, 2010). That said, there remains to be seen whether or not the standards written within the CCSS will give the United States the competitive edge because, as of yet, no student has completed a full k-12 sequence of CCSS. Therefore, it cannot be shown that it will make students more competitive in the global economy, nor can it be shown that these standards will make students college and career ready. This curriculum project, however, will do its best to follow the CCSS as written and emphasize mathematical modeling and the use of real-world problems to try to prepare students for real world thinking.

There are five common reasons identified for the inclusion of real world problems in the mathematics classroom. These are: (1) utilitarian purposes to help show students how to apply mathematics to vocational work outside the classroom, (2) using mathematics to teach students about important issues, (3) improving understanding of concepts, (4) increasing appreciation of the nature of mathematics, and (5) Improving student affect toward mathematics (Beswick, 2010). Research remains mixed as to whether these last three of the five reasons can be substantiated with evidence (Beswick, 2010), but to meet the higher emphasis on mathematical modeling, hands-on student activities and application problems have been included in this curriculum project.

**Previous Standards for Mathematics Education**

The National Council of Teachers of Mathematics (NCTM) created the first set of standards in 1980, The Principles and Standards of School Mathematics. They precede the CCSS, but no incentive was given for states to adopt these standards. The Mathematical Association of America (MAA), the national organization of mathematicians and mathematics professors, provided suggestions and recommendations to the NCTM about the Standards. These concerns were addressed and the NCTM Standards were updated and published again in 1989.
The most recent update of the NCTM Principles and Standards was in 2000 (NCTM, 2000). The NCTM standards include standards of communication and connections to problems outside the mathematics classroom (NCTM 2000), but perhaps without as much emphasis as they get in the CCSS.

A proposed 2.5 billion dollars is to be purposed to help states align curricula to the CCSS, and such funding and monetary incentives has helped the CCSS gain such a high rate of adoption (Mathis, 2010). This money comes from the Race to the Top funds which were created as a part of the Recovery and Reinvestment Act of 2009. The states that choose to implement specific changes in their state education programs are rewarded with monetary funding (U.S. Department of Education, 2009). These specific changes include (1) the adoption of common core standards, (2) the creation of a data system to measure student growth and success, (3) recruiting, developing, rewarding, and retaining effective teachers and principals, and (4) turning around the lowest-achieving schools (U.S. Department of Education, 2009). Currently, 45 states have adopted and will all be following the CCSS- all of which will now be required to implement curricula that emphasizes real-world situations and modeling (National Governors Association Center for Best Practices, 2010).

**CCSS Standards and Curriculum in Trigonometry**

For the purpose of this paper, standards are defined as the set of expectations and goals—what a student should be able to know and do. Curriculum is defined as a detailed daily plan for teaching including the lessons, activities, and assessments used for learning. The CCSS establish a set of standards for teaching mathematics and English; they are *not* a curriculum and therefore, it will be up to the individual states, schools, and teachers to interpret and implement the standards (Rust, 2012). In New York State, EngageNY uses the RTT funding to provide
assistance with instructional resources, teacher effectiveness materials, and professional
development (EngageNY, 2013). Even so, teachers may still find it difficult to decipher what
defines a classroom lesson that adheres to the CCSS.

It is generally agreed upon that teachers of mathematics would like students to
understand the mathematical concepts they are applying in their mathematics classes, but several
researchers have observed that many trigonometry classrooms are predominantly focused on
procedural learning (Weber, 2005). The trigonometry standards written in the CCSS seem to
require a deeper understanding of the topic. The unit contained within this curriculum project
may be useful for high school trigonometry teachers who are seeking ideas for teaching a unit to
introduce the trigonometric functions and radian measures connected to the unit circle in a way
aligns with the new CCSS.

The trigonometry required in this curriculum project requires students to have previously
developed an understanding of trigonometry as the ratios of pairs of sides of a right triangle. This
unit develops the trigonometric relationships as a function on an angle using the unit circle.
Students will be introduced to radian measure, learn to graph basic trigonometric functions, and
use these functions to model real-world situations. These lessons will fulfill two content
standards written within the CCSS standards for high school functions:

CCSS.Math.Content.HSF-TF.A: Extend the domain of trigonometric functions using the
unit circle.

CCSS.Math.Content.HSF-TF.B: Model periodic phenomena with trigonometric functions
(National Governors Association Center for Best Practices, 2010).

This unit may be best suited for a second year algebra or trigonometry course often associated
with the tenth or eleventh grade. Historical and current teaching practices of trigonometry have
been considered throughout this research to design a unit that is beneficial to student understanding and retention.
Chapter 2: Literature Review

According to the authors of the CCSS, there are eight standards for mathematical practice. These are standards that mathematics teachers should instill in their students across all of the content standards. These are the standards that a good curriculum should contain. The eight standards are as follows: (1) that a student can make sense of problems and persevere in solving them; (2) reason abstractly and quantitatively; (3) construct viable arguments and critique the reasoning of others; (4) model with mathematics; (5) use appropriate tools strategically; (6) attend to precision; (7) look for and make use of structure; and (8) look for and express regularity in repeated reasoning (National Governors Association Center for Best Practices, 2010). Many of these standards for mathematical practices come in response to its predecessors. First written in 1989, and last revised in 2000, the NCTM published its Principles and Standards for School Mathematics which included five process standards: problem solving, reasoning and proof, communication, representation, and connections (NCTM, 2000). These five NCTM standards were used by the authors of the CCSS to create the first few standards for mathematical practices in the CCSS (National Governors Association Center for Best Practices, 2010). Since the adoption of the CCSS by 45 of the 50 states, it is now important that teachers in the 45 states align their curriculum with the standards of the CCSS. Key features of the CCSS for mathematics grades 9-12 include a focus on applying mathematical thinking to real-life situations that prepare students to think mathematically, and the use of mathematical modeling and statistics to analyze empirical situations and improve decision making (Rust, 2012).

With these standards in mind, this curriculum project intends to build a trigonometry curriculum that will address these standards. First, the curriculum project will examine what students should know about the basic trigonometric functions according to the standards, and
then it will consider some of the historical ways trigonometry has been taught before developing
a new curriculum that will unify old and new methods in a way that will help students model and
connect to the topic.

**Sequencing a Trigonometry Curriculum**

A typical American high school will usually teach trigonometry in a third year
mathematics course following algebra and geometry. It introduces students to families of
functions content with many practical applications including applications in surveying,
arquitectura, Newtonian physics, and many engineering fields (Weber, 2005). The CCSS in
mathematics has prepared suggestions for the implementation of its content standards across four
years of high school. The content standards that relate primarily to teaching trigonometry fall
into two categories of the high school mathematics standards: geometry, and functions. The
CCSS in mathematics suggests a sequence of the items pertaining to trigonometry shown in
Table 1. The curriculum that follows will address CCSS.Math.Content.HSF-TF.A.1,
CCSS.Math.Content.HSF-TF.A.2, CCSS.Math.Content.HSF-TF.A.3 and
CCSS.Math.Content.HSF-TF.A.5 in the table below.

There are often two methods to introducing trigonometry to students. One is the ratio
method of triangle trigonometry in which students consider the trigonometric functions as the
ratios of pairs of sides of a right triangle, and by the other method, students learn the
trigonometric functions through analysis of the unit circle (Kendal, 1996). Table 1 seems to
suggest that the CCSS in mathematics recommends that students first learn by the ratio method,
and then expand their understanding to the circle method in the subsequent years. Trigonometric
functions often pose a new type of problem to students- they are some of the first functions
students see that cannot be directly evaluated through algebraic methods. The skills developed
<table>
<thead>
<tr>
<th>Course</th>
<th>Geometry</th>
<th>Algebra 2</th>
<th>Fourth Year Course</th>
</tr>
</thead>
<tbody>
<tr>
<td>Common Core Standard</td>
<td>Define trigonometric ratios and solve problems involving right triangles.</td>
<td>Extend the domain of trigonometric functions using the unit circle.</td>
<td>Extend the domain of trigonometric functions using the unit circle.</td>
</tr>
<tr>
<td></td>
<td>• CCSS.Math.Content.HSG-SRT.C.6 Understand that by similarity, side ratios in right</td>
<td>• CCSS.Math.Content.HSF-TF.A.1 Understand radian measure of an angle as the length of the arc on the unit circle</td>
<td>• CCSS.Math.Content.HSF-TF.A.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for π/3, π/4 and π/6, and use the unit circle to express the values of sine, cosine, and tangent for x, π + x, and 2π – x in terms of their values for x, where x is any real number.</td>
</tr>
<tr>
<td></td>
<td>triangles are properties of the angles in the triangle, leading to</td>
<td>subtended by the angle.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>definitions of trigonometric ratios for acute angles.</td>
<td>• CCSS.Math.Content.HSF-TF.A.2 Explain how the unit circle in the coordinate plane enables the extension of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• CCSS.Math.Content.HSG-SRT.C.7 Explain and use the</td>
<td>trigonometric functions to all real numbers, interpreted as radian</td>
<td></td>
</tr>
<tr>
<td></td>
<td>relationship between the sine</td>
<td>measures of angles traversed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>and cosine of complementary</td>
<td>counterclockwise around the unit circle</td>
<td></td>
</tr>
<tr>
<td></td>
<td>angles.</td>
<td>Model periodic phenomena with trigonometric functions.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• CCSS.Math.Content.HSG-SRT.C.8 Use trigonometric ratios and the</td>
<td>• CCSS.Math.Content.HSF-TF.B.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Pythagorean</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Theorem to solve right triangles in applied problems.</td>
<td>Prove and apply trigonometric</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Apply trigonometry to general</td>
<td>identities.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>triangles</td>
<td>• CCSS.Math.Content.HSF-TF.B.6 (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• CCSS.Math.Content.HSG-SRT.D.9 (+) Derive the formula A = 1/2 ab sin(C)</td>
<td>Prove and apply trigonometric</td>
<td></td>
</tr>
<tr>
<td></td>
<td>for the area of a triangle by drawing an auxiliary line from a vertex</td>
<td>identities.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>perpendicular to the opposite side.</td>
<td>• CCSS.Math.Content.HSF-TF.C.8 Prove the Pythagorean identity sin²(θ) + cos²(θ) = 1 and use it to find sin(θ), cos(θ), or tan(θ) given sin(θ), cos(θ), or tan(θ) and the quadrant of the angle.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• CCSS.Math.Content.HSG-SRT.D.10 (+) Prove the Laws of Sines and Cosines</td>
<td>Prove and apply trigonometric</td>
<td></td>
</tr>
<tr>
<td></td>
<td>and use them to solve problems.</td>
<td>identities.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• CCSS.Math.Content.HSG-SRT.D.11 (+) Understand and apply the Law of</td>
<td>• CCSS.Math.Content.HSF-TF.C.9 (+) Prove the addition and subtraction</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sines and Cosines and use them</td>
<td>formulas for sine, cosine, and tangent and use them to solve problems.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>to solve problems.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• CCSS.Math.Content.HSG-SRT.D.11 (+) Understand and apply the Law of Sines and Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(National Governors Association Center for Best Practices, 2010)

through trigonometry are skills necessary for success in upper level math courses such as pre-calculus and calculus (Weber, 2008), and simultaneously links together algebraic, geometric
and graphical thinking (Weber, 2005). There remains little research in student learning and understanding of trigonometry despite the records of student difficulties in this topic (Weber, 2005). It may be worthwhile to consider the origins of the study of trigonometry when it comes to determining methods of teaching trigonometry in a classroom of mathematics.

**Historical Foundations of Trigonometry**

Many believe that it is necessary for students to first study triangle trigonometry and the ratios of sides of right-angled triangles before studying circle trigonometry (Bressoud, 2010). It is, however, a historical fact that trigonometry in its earliest form was circle trigonometry and arose from classical Greek study of the heavens. It was not until over a thousand years later that triangle trigonometry developed from questions of finding heights of objects from the shadows they cast (Bressoud, 2010). This historical analysis can become the foundation of the argument of first teaching students circle geometry before learning triangle geometry. That said, students are very often taught this topic in the reverse order- first focusing on triangular ratios, and then discussing the unit circle (Weber, 2008). The CCSS pathways in the appendix A of the standards suggests that students first learn trigonometric ratios and then learn about trigonometry with respect to the unit circle (National Governors Association Center for Best Practices, 2010).

**Teaching Trigonometry: Ratios vs. The Unit Circle**

Of these two methods of beginning our study of trigonometry, the right-triangular ratios method proves to be most popular in many high school textbooks (Weber, 2008). Students often memorize the mnemonic device SOHCAHTOA under this method- a mnemonic that reminds students that $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, and $\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$. In a study comparing the two methods of learning analyzed by Kendal and Stacey, students were given pre- and post- assessments to test student ability to identify missing sides of right triangles, and
solving equations with decimals. The results of the study showed that students who learned through the ratios method attained higher levels of accuracy and had better retention of knowledge as opposed to students who learned the material through the unit circle (Kendal, 1996). Students were more likely to make errors in setting up reference triangles on the unit circle (Kendal, 1996). This approach necessitated that students memorized more procedure to solve missing-side problems (Kendal, 1996). Students also seemed to respond more favorably to the ratios teaching method. When surveyed on their liking of trigonometry on the pre- and post-tests, students in the classroom emphasizing trigonometric functions as ratios saw a greater improvement than the unit circle classes (Kendal, 1996). This discrepancy could be linked to the fact that students who learned to identify missing sides of a right triangle with the unit circle method had to memorize a longer procedure with more room for error built in.

In 1989, the National Council of Teachers of Mathematics (NCTM) Standards advocated for a change in the way teachers of mathematics traditionally taught trigonometry with methods that focus on “memorization of isolated facts and procedures and by proficiency with paper-and-pencil skills” (NCTM, 1989). Triangle trigonometry, if taught by memorization of ratios like sohcahtoa, seems counter to the goals established by NCTM. Further, the 2000 NCTM Standards emphasized student ability to interpret functions and the importance of technology in teaching and learning mathematics (NCTM 2000). The unit circle approach makes it easier for students to visualize trigonometric functions as functions of an angle. Students can more readily see on a unit circle that the input is a unique angle, and the output is the length of a segment created by a unique triangle within the unit circle (Weber, 2008). In his 2005 study, Weber observed that students enrolled in a college trigonometry class who were given the task of determining sines and cosines by drawing out unit circles and reference angles had a better conceptual
understanding of sine and cosine as functions of an angle. For example, students were asked to predict which would be a larger value: \( \sin 23^\circ \) or \( \sin 37^\circ \). They were expected to sketch and/or visualize a unit circle to solve the problem. Similarly, students were also asked to justify why \( \sin \theta = 2 \) is an impossible value. Students in classes that emphasized conceptual understanding of trigonometric functions had greater success rates at providing proper justifications to the two preceding problems (Weber, 2005). The unit circle method also downplays the importance of the rote memorization of triangle proportions and allows students to concretely view trigonometric functions as line segments in a circle of radius 1 (Bressoud, 2010).

**When to use Technology**

Many sources that discuss the teaching of trigonometry advocate for supplementing trigonometry with technology. Technology and calculator use has significantly decreased the amount of time spent in the classroom looking up values on trigonometric tables and present teachers with a great tool that allows students to take a laboratory approach to learning mathematics; students can use data and software to model real-world periodic phenomena (Hirsch, 1991). In one case study, Sang (2003) identified that a benefit of teaching trigonometry through technology is that it encouraged his student to “get over his passive learning attitude toward mathematics,” and he learned to be in charge of his own learning. In trigonometry, technology in the classroom could make it easier for students to explore families of functions. Technology can assist a student in investigating and understanding graphs such as \( f(x) = a \sin(bx - c) + d \) as transformations on the parent function \( f(x) = \sin x \). Students can relatively quickly discern through experimentation what effect different values of \( a, b, c \) and \( d \) have on the original parent function, graphing many graphs in a short amount of time (Hirsch, 1991). However, some teachers may be concerned that the use of calculators will lead to a complete
reliance upon them to the point where students are not competent without them. Additionally, calculators introduce a new set of problems to learning. Though there are numerous positive outcomes, graphing calculator technologies also requires students to become particularly conscious of different viewing rectangles. While the graph of $y = \sin x$ may appear to change as the viewing window is manipulated on a calculator, in the end, it is up to the students to realize no real change has occurred in the shape of the graph (Hirsch, 1991).

An additional issue of technology in the classroom to learn trigonometry is the sequence in which this should occur. That is, is it more beneficial to first receive classroom instruction then explore with technology, or is it better to introduce trigonometry with technology before beginning whole-class teaching? A study by John Ross, Catherine Bruce and Timothy Sibbald investigated into these two different sequences and found that students learned more when they received their technology instruction after full-class instruction as opposed to the other way around. However, by their post-test, there were no significant statistical differences between the two groups (Ross, 2011).

**Student Perceptions**

Many students believe mathematics to be primarily verbal and formulaic in nature. The unit circle diagram method of introducing trigonometry to students, though not heavily dependent upon technology, promotes visual and spatial thinking (Shear, 1981). Students who identify themselves as visual learners may find great benefit in sequencing the learning of trigonometry to focus on the unit circle first, and technology later. The technique in Weber’s 2005 study deemphasizes the use of technology and instead favors pencil-and-paper drawings of the unit circle and reference angles (Weber, 2005). This curriculum project followed a similar strategy in deemphasizing the use of technology at first to give students the opportunity to
construct their own models and understandings of the basic trigonometric functions. It would then be suggested to follow up with technology in a way as indicated by Hirsch (1991) in later lessons.

Trigonometry prompts students to link together many areas of mathematics from numerical relationships to graphical reasoning. Trigonometric functions are some of the first functions students encounter that they are not able to evaluate directly through algebraic operations, and therefore require a deeper conceptual understanding (Weber, 2008). It is these reasons among many reasons that students may struggle in their understanding of this topic. Teachers must similarly struggle to find the methods best suited to their students. Research leads us down several paths that ask us to balance unit circle methods against proportions in triangles and technology use against paper-and-pencil skills.
Chapter 3: Introduction to Trigonometric Functions Unit Plan

The following unit plan was designed for use in an Algebra 2/ Trigonometry mathematics classroom. In many states, students traditionally take this class in their third year of high school, but these lessons could be applicable anywhere that students are being introduced to radian measures and connecting the unit circle to trigonometric functions. Students will learn to connect the concept of a trigonometric function as a function on an angle. The paper plate activities given within these lessons are modified from materials available at (http://nctm.confex.com/nctm/2012IL/webprogram/Session11360.html). All of the lessons are originally built to spend about 50 minutes on the lesson.

Unit Timeline

Day 1- What is a radian?
Day 2- Radian and degree conversions and position on the coordinate plane
Days 3- Identifying unit circle values using reference triangles
Day 4- Graphing Sine and Cosine curves with Pasta
Day 5- The Tangent Curve
Day 6- Periodic functions- period and amplitude (And a unit circle quiz)
Day 7- Periodic Function applications
Day 8- Review
Day 9- Test

Unit Standards

The following CCSS will be covered within this unit

Extend the domain of trigonometric functions using the unit circle.
CCSS.Math.Content.HSF-TF.A.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
CCSS.Math.Content.HSF-TF.A.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

CCSS.Math.Content.HSF-TF.A.3 Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $x$, $\pi + x$, and $2\pi - x$ in terms of their values for $x$, where $x$ is any real number.

**Model periodic phenomena with trigonometric functions.**

CCSS.Math.Content.HSF-TF.B.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.
Day 1 Lesson- What is a radian?

Objectives: Students will make connections between degrees and radians and learn that the radian measure of an angle is the length of the arc on the unit circle subtended by the angle.

Learning Standards (CCSS): CCSS.Math.Content.HSF-TF.A.1

Materials:
- Lesson Worksheet
- Flat paper plates (1 per student)
- Scissors
- String

Lesson Procedure:

First it is important for students to realize the fact that there are 360 degrees in a full rotation of a circle is a somewhat arbitrary number that dates back to historical times. A quick internet search will yield reasons such as 360 being close to the number of days in a year, or cite that it is a convenient, highly divisible number. You may have students think about and briefly discuss the following question:

Why does a circle have 360 degrees? (Who decided that number should be 360 and not some other number?)

Allow students time to answer, but it is important for them to see there is no apparent reason why we have to use 360.

Introduce radians as a different way to measure an angle using the following guided activity. Students will receive the following worksheet and follow the procedure to answer the questions included.

The conclusion will be discussed as a class: There are π radians in a semicircle and 2π radians in a circle.

With the remaining time, students will be given two challenge questions to work on and turn in as they leave the classroom. They are as follows:

- How many radians are there in a quarter of a circle (90 degrees)?
- How many radians in three quarters of a circle (270 degrees)?
What is a Radian?

1. Take your paper plate and fold it in half. You have just found the diameter of your plate!

2. Cut a piece of string equal to the length of the diameter of your circle.

3. Measure around the outside of your plate with the piece of string.
   
   a. The outside of the circle measures just over _____ lengths of the piece of string.
   
   b. The formula for the circumference of a circle is \(C = \pi D\) where \(D\) is the diameter of the circle. How does this formula connect to your answer in part (a)?

   __________________________________________________________
   __________________________________________________________
   __________________________________________________________

4. Now fold your plate in half again to form some axes. Use your pencil to draw in these axes. Label the center \(O\). Label a point on the axes along the rim of the circle \(A\) (see diagram).

5. Cut a piece of string to the length of the radius of the circle. Starting at \(A\), measure along the rim of the circle as far as the string will go. Label the other endpoint of the string \(B\), and draw in angle \(\angle AOB\) (see diagram).

*** The measure of the central angle \(\angle AOB\) whose arc length is 1 radius is one radian.***

6. Estimate the number of degrees in one radian. Record your guess. _______
   Now use a protractor to measure the angle. Record the measured value. _______

7. Now estimate the number of radians in a semicircle. Starting at point \(B\), use the radius string to continue to measure out angles of 1 radian. Number of radians in a semicircle: _______

   Conclusion: There are ______ radians in a semicircle and ______ radians in a circle.
Day 2 Lesson- Radian and degree conversions, and position on the coordinate plane

Objectives: Students will know how to convert between degrees and radians by means of proportions (and the fact that $\pi = 180^\circ$). Students will label the quadrants of the coordinate plane and be able to draw the terminal sides of angles in radian measure into the coordinate plane.

Learning Standards (CCSS): CCSS.Math.Content.HSF-TF.A.1

Materials:

- Lesson Worksheets

- Display copy (smartboard, or overhead or document cam)

- Colored pencils in red, green, and blue

Lesson Procedure:

Using the lesson worksheet on the following page, the first page will be used as guided notes. Students will learn how to set up a proportion to make conversions, and also learn that to convert radians to degrees, they can substitute 180 where they see the $\pi$ symbol.

They will complete the second page of examples either independently or in pairs.

The third page will be a review of the coordinate plane and will help students become more comfortable identifying angles in radian measure.
Name____________________
Day 2 Notes

**Degrees ↔ Radians**

Last class, we learned that there are _______ radians in a semicircle,
and a semicircle is ______ degrees.

Use this knowledge to set up a proportion. \( \frac{\text{Degrees}}{\text{Radians}} \)

Because degrees and radians are proportional, we can use this ratio and set up a proportion. Do this to convert 30° into radians.

Use a proportion to convert \( \frac{4\pi}{3} \) radians into degrees.

Can you think of an easier way to convert radians to degrees? Try it with the same problem above.
Now try these on your own! Convert from degrees to radians. Reduce fractions to simplest form.

a.  $-32^\circ$  

b.  $215^\circ$  

c.  $83^\circ$  

d.  $400^\circ$

Convert the following radian measures back to degrees.

a.  $\frac{\pi}{9}$  

b.  $\frac{3\pi}{4}$  

c.  $7\pi$  

d.  $-\frac{5\pi}{4}$

Usually, we don’t put any units on radial measures. What’s one way we might be able to tell if an angle is measured in radians?
Recall: Each quadrant of the coordinate plane is given a number. We label these I, II, III, and IV. They are labeled in a counterclockwise direction. Label the set of axes below.

Definitions:

Initial side of an angle - the side that the measurement of an angle starts from

Terminal side of an angle - the side at which the measurement of an angle ends

Normally we place the initial side along the positive x axis and call this 0°. We measure angles in a counterclockwise direction.

Draw each of the following angles into the axes above

- Label the positive x-axis 0 for zero degrees or radians.
- Determine the radian measures for 90°, 180°, 270°, and 360°, and place these measures where they belong on the axes above.
- In RED, draw an angle whose terminal side is in quadrant III.
- In GREEN, draw the terminal side of the angle whose measure is $\frac{5\pi}{6}$. In which quadrant does this angle end? __________
- In BLUE, draw the terminal side of the angle whose measure is $-\frac{\pi}{4}$. In which quadrant does this angle end? __________
Day 3- Identifying unit circle values using reference triangles

Objectives: Students will gain an introduction to the unit circle and be able to determine the coordinates of certain points along the circle using reference triangles

Learning Standards (CCSS): CCSS.Math.Content.HSF.TF.A.3

Materials:
- Flat paper plates. One for each student.
- Card stock triangles in different colors cut into 45-45-90, 30-60-90 and 60-30-90 right triangles (yes, even though two of these are the same, it is important to do these differently) 2 each for every student. The hypotenuse of each triangle should be equal to the radius of the plate
- Tape
- A blank unit circle for students to fill in such as the one available at (http://www.gradeamathhelp.com/unit-circle.html)

Lesson Procedure:

Warm-up:

Draw a 45-45-90 and a 30-60-90 right triangle on the board with a hypotenuse length of 1. As a warm up, ask students to find the lengths of the missing sides using trigonometry or to recall what they know about special right triangles. Optional: provide students with the formulas

\[
\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \text{and} \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}
\]

on the board with the problem.

Discuss the results and review the special right triangle ratios.

Lesson body:

Define a unit circle (this may be written on top of their unit circle handout page)

Unit Circle- A circle with a radius of 1 unit centered at the origin

Students will receive a new paper plate, two of each reference triangle (45-45-90, 30-60-90 and 60-30-90) and a blank unit circle to fill in the missing coordinates and radian measures.

Students will begin by folding their plates in quarters and drawing in the axes. Students will label \(0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \text{ and } 2\pi\) radians. As a class, help students fill out the angle measures on the unit circle page all the way around the circle.
Tape the triangles on the plate with right angles along the x-axis in such a way that they can flip over the axis and be used for reference triangles in quadrants I and IV and in quadrants II and III (see Figure 1).

Figure 1. Reference triangles on a paper plate unit circle.

Label the lengths of the sides of each triangle with the appropriate sign (positive or negative) then flip the triangles across the axis and do the same to the other side.

Now they can label the coordinates on their unit circle page using the lengths on their triangles. Teacher may need to assist in this process.
Day 4- Graphing sine and cosine functions

Objectives: Students will gain an introduction to the unit circle and be able to determine the coordinates of certain points along the circle using reference triangles


Materials:
- A unit circle
- String
- Scissors
- Markers
- Linguini noodles
- Butcher paper
- Tape

Lesson Procedure:

Warm-up: Students will receive a blank unit circle. Ask them to fill out the angle measures in radians of the marked angles all around the circle.

Lesson body: Students will work in groups of 2-4. Half the groups will be assigned the cosine curve; the other half will be assigned the sine curve.

Students will use the lab notes to complete the graphing activity. This lab idea is credited to Lisa Huddleston, Mathematics Teacher and Academic Dean, Raleigh Charter High School (http://www.raleighcharterhs.org/aboutus/racg/Lab-SpaghettiTrig.pdf)

Conclusion: Students will compare what they see between groups (both sine and cosine curves) and draw some conclusions. You may wish for students to write down their findings and hand these to you on their way out the door.
My group will be graphing the ___________ function.

(sine or cosine?)

Since we are using the unit circle, you will either be graphing the x, or y values
of the coordinates around the unit circle. Remember,
\[ x = \cos \theta \]
\[ y = \sin \theta \]

**Gather the following materials for your group**
1 piece of butcher paper, several colored markers, 1 blank unit circle, 1 piece of yarn,
10 strands of spaghetti, 1 yard stick, and 1 roll of tape

**Procedure:**

1. Place one end of your yarn at the point (1, 0) on your unit circle. Carefully wrap the yarn
counterclockwise around the circumference of your unit circle. Now use a marker to
mark points along the string that correspond to each special radian measure indicated on
your unit circle: \(0, \pi/6, \pi/4, \pi/3, \pi/2\), etc. Mark the yarn up to the value \(2\pi\). You may
need to tape the yarn down to make this step easier.

2. You will be graphing either the function \(f(\theta) = \sin \theta\) or \(f(\theta) = \cos \theta\). See above for
which function your group has been assigned. This means the horizontal (independent)
axis measures the angle \(\theta\), and the vertical (dependent) axis measures your trig value
\(f(\theta)\). On the butcher paper, use your yard stick to draw and label the y-axis vertically
down the center of the paper. Then, draw and label the x-axis as well.

3. Take your marked yarn off of the unit circle, and place the end that corresponded to the
point (1,0) on the unit circle at the origin and stretch the yarn across the positive
horizontal axis which measures the angle \(\theta\). The marks you made on the yarn correspond
to given angle measures. Therefore, the first mark on the yarn can be transferred onto the
horizontal axis on your butcher paper and labeled \(\pi/6\). The second mark can be transferred
and labeled \(\pi/4\) and so on until all the points up to \(2\pi\) have been marked on the paper.
4. Now that the positive horizontal axis is marked, flip the yarn over (reflect it across the vertical axis) and mark the negative horizontal axis using the same method as in step 3. The first point this time will correspond to the angle $-\frac{\pi}{6}$.

Complete only one of the two steps below: Those graphing sine will use 5a, those graphing cosine will use 5b.

5a. For those graphing sine function only: For each of the reference angles you identified ($0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, etc.), there is a corresponding right triangle (reference triangle) you can draw. Since you are graphing sine, you will be graphing the y value of each of these triangles. You can think of this as measuring vertical side of each reference triangle. Instead of using a ruler to measure the length of the vertical side, you will be using spaghetti. Take your spaghetti and place one end on the horizontal axis and the rest along the vertical side of your reference triangle. Break the spaghetti off at the point where the vertical side of the reference triangle intersects with the unit circle. Now take this length of spaghetti and tape it to your butcher paper at the corresponding reference angle. You will repeat this process for each reference angle around the unit circle. Keep in mind that when you get to the third and fourth quadrants, you are measuring down, so you should tape your spaghetti below the horizontal axis.

5b. For those graphing cosine function only: For each of the reference angles you identified ($0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, etc.), there is a corresponding right triangle (reference triangle) you can draw. Since you are graphing cosine, you will be graphing the x value of each of these triangles. You can think of this as measuring horizontal side of each reference triangle. Instead of using a ruler to measure the length of the horizontal side, you will be using spaghetti. Take your spaghetti and place one end on the origin and the rest along the horizontal side of your reference triangle. Break the spaghetti off where the horizontal side of the reference triangle ends. Now take this length of spaghetti and tape it to your butcher paper vertically at the corresponding reference angle. You will repeat this process for each reference angle around the unit circle. Keep in mind that when you get to the second and third quadrants, you are measuring in the negative direction, so you should tape your spaghetti below the horizontal axis.

6. Once all your spaghetti is taped down, ask your teacher to check your graph. You will then take a marker to connect the ends of the spaghetti with a smooth curve. Label it either $f(\theta) = \sin \theta$ or $f(\theta) = \cos \theta$ depending on which one you were assigned.

CONGRATULATIONS! YOU HAVE GRAPHEP YOUR TRIG FUNCTION USING SPAGHETTI!
Figure 2. Blank unit circle for classwork.

The lines of intersection around the circle above represent the radian measures $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{7\pi}{4}, \frac{11\pi}{6}$. 
Day 5- The Tangent Curve

Objectives: Students will evaluate the values of tangent for angles around the unit circle. Students will use these values to create a graphical representation of the tangent function both on the x-y plane and as a segment of the unit circle. Students will identify the quadrant in which each trigonometric function is positive/negative.


Materials:
- Student note sheet
- Class copy of notes (projector, smartboard, etc)

Lesson Procedure:

Warm-up: Students will receive the note sheet for the day at the beginning of class. The warm up activity is to use their prior knowledge (and notes) of the unit circle to help fill in the values of sine and cosine for all the identified angles around the unit circle. Students will not fill out the tangent row during the warm-up.

Lesson body: Have students recall that \( \tan \theta = \frac{\sin \theta}{\cos \theta} \). Depending upon students’ background knowledge, this may require a simple proof (It is assumed students already know SOHCAHTOA or some other equivalent). Use this fact to then guide students to calculate all the table values of tangent on the chart. Students may need help identifying that dividing by zero means the function will be undefined.

Demonstrate what tangent looks like on a unit circle. The teacher may wish to draw out several diagrams for multiple examples. A completed diagram may look something like the one pictured below in Figure 3:

![Diagram of unit circle with sine, cosine, and tangent segments labeled](image_url)

Figure 3. Identifying sine cosine and tangent segments on a unit circle
Next, students will complete graphs of the three functions. Students may need guidance in how they can use their table of values to find the negative values of $\theta$. They have already seen the sine and cosine graph in the previous day’s lesson.

Conclusion: Students will use the graphs to answer the remaining questions on the note sheet.
Sine, Cosine, and Tangent

Use your Unit Circle to help fill out the following chart for sine values and cosine values of \( \theta \). Don’t worry about the tangent row for now!

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0</th>
<th>( \pi/6 )</th>
<th>( \pi/4 )</th>
<th>( \pi/3 )</th>
<th>( \pi/2 )</th>
<th>( 2\pi/3 )</th>
<th>( 3\pi/4 )</th>
<th>( 5\pi/6 )</th>
<th>( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0°</td>
<td>30°</td>
<td>45°</td>
<td>60°</td>
<td>90°</td>
<td>120°</td>
<td>135°</td>
<td>150°</td>
<td>180°</td>
</tr>
<tr>
<td>sin ( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>cos ( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tan ( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( 7\pi/6 )</th>
<th>( 5\pi/4 )</th>
<th>( 4\pi/3 )</th>
<th>( 3\pi/2 )</th>
<th>( 5\pi/3 )</th>
<th>( 7\pi/4 )</th>
<th>( 11\pi/6 )</th>
<th>( 2\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>210°</td>
<td>225°</td>
<td>240°</td>
<td>270°</td>
<td>300°</td>
<td>315°</td>
<td>330°</td>
<td>360°</td>
<td></td>
</tr>
<tr>
<td>sin ( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>cos ( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tan ( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

You may recall \( \tan \theta = \frac{\sin \theta}{\cos \theta} \). Use this fact to calculate the value of \( \tan \theta \) for all \( \theta \) above. Use the space below for any necessary calculations.
We know how sine and cosine relate to the unit circle, but what does tangent look like? Draw it in below:

Now let’s graph our three trigonometric functions in the spaces provided below:

\[ f(\theta) = \sin \theta \]

\[ f(\theta) = \cos \theta \]
Questions:

1. What do you think will happen if we go beyond $2\pi$ in either direction on the x-axis? Why?

2. State the domain and range of each function:

   \[ f(\theta) = \sin \theta \quad \text{Domain:} \quad \text{Range:} \]

   \[ f(\theta) = \cos \theta \quad \text{Domain:} \quad \text{Range:} \]

   \[ f(\theta) = \tan \theta \quad \text{Domain:} \quad \text{Range:} \]

3. Is it possible for $\sin \theta = 5$? Explain why or why not.

4. Is it possible for $\tan \theta = 5$? Explain why or why not.
Day 6 - Sine, Cosine, and Tangent: Calculating using reference triangles

Objectives: Students will use reference triangles to calculate exact values of sine, cosine, and tangent for angles greater than 90° and greater than 360°.


Materials:

- Paper plates from Day 3 lesson
- Student note sheets
- Class copy of notes (projector, or smartboard, etc.)

Lesson Procedure:

The purpose of this lesson is to strengthen the students’ skills in using the unit circle, and mastering the use of reference triangles to find exact values of \( \theta \) greater than \( \frac{\pi}{2} \).

Warm-up: Students will complete a warm up, shown on the following page, that will ask them to identify the sign (positive or negative) of the three basic trigonometric functions in each quadrant.

After discussing the warm up and discussing the students’ possible memory devices, hand out the notes. Share ASTC (All Students Take Calculus) with them. Remind them that they don’t need the mnemonic. If they remember how to use the unit circle, they can determine where sine and cosine are positive and negative using the lengths of their corresponding segments in each quadrant. The point of the mnemonic is to speed up this process- not to mindlessly memorize.

Using the paper plates, students should notice that, if asked to identify every location where sine takes on an absolute value of \( \frac{1}{2} \), each of these four locations uses the same reference triangle. Namely, the 30-60-90 right triangle. The teacher will explain that this triangle is what we call the reference triangle and we can use the reference triangle to evaluate the exact value of sine, cosine, or tangent in any quadrant for multiples of these angles in the reference triangle.

As a class, go over the next two examples together. Students may use their plates to verify their answers. Students will complete the rest of the sheet independently while teacher observes and answers individual questions before going over the answers together as a class.

NOTE: The following class day will include a quick quiz on the unit circle (identifying coordinates around the unit circle)
Warm-up:

Label the quadrants (I, II, III, and IV) of the coordinate plane below. In each quadrant state whether each of the three listed trigonometric functions is Positive (+), or Negative (−). You may use yesterday’s notes to do so.

Try and look for patterns. What memory device can you come up with to remember where each trig function is positive?
Using Reference Triangles to Find Exact Trig Values

To remember where Sine, Cosine, and Tangent are positive, we can use the mnemonic ASTC (All Students Take Calculus).

- In Quadrant I, All three trig functions are positive.
- In Quadrant II, Sine is positive, the other two are negative.
- In Quadrant III, Tangent is positive, the other two are negative.
- In Quadrant IV, Cosine is positive, the other two are negative.

Using your paper plate, identify all locations where \(|\sin \theta| = \frac{1}{2}\) (the absolute value of sine)

This should happen 4 times. Each time, you should notice we use a 30-60-90 right triangle in each of the four quadrants. These are Reference Triangles. You make a reference triangle by dropping a perpendicular from the terminal side of the angle to the x-axis.

Draw a reference triangle for \(\frac{7\pi}{6}\).

What is the central angle in the reference triangle you drew?

We call the above angle the Reference Angle. Calculate the sine of this reference angle.

Is sine Positive or Negative in Quadrant III?

What is \(\sin \frac{7\pi}{6}\)?
Now use reference triangles to calculate $\cos \frac{11\pi}{4}$. Include a picture of the reference triangle in your solution. (Tip: If you are ever struggling with radian measure, you can always convert it into degrees)

Try these on your own:

a. $\tan \frac{4\pi}{3}$

b. $\cos \frac{11\pi}{3}$

c. $\sin \frac{7\pi}{2}$

d. $\cos \frac{13\pi}{4}$

**Conclusion:** Reference triangles allow us to find the exact value of multiples of the angles $\frac{\pi}{6}, \frac{\pi}{4},$ and $\frac{\pi}{3}$ by knowing trig values for these three values and some simple facts about the signs of the functions in the associated quadrant.
Day 7- Unit Circle Quiz & Part 1 of Trig Transformations

Objectives: Students will be able to identify coordinates of points around the unit circle. Students will recognize the difference between periodic and non-periodic functions and be able to define amplitude and period, and calculate these two values given the images of various periodic functions.

CCSS.Math.Content.HSF-TF.B.5

Materials:

- Student quizzes

- Handouts of the notes page

- Display copy of notes

(images of periodic functions from the notes come from the textbook Prentice Hall Mathematics: Algebra 2 by Allan E. Bellman, Sadie Chavis Bragg, Randall I. Charles, Basia Hall, William G. Handlin, Sr., and Dan Kennedy)

Lesson Procedure:

Students will take the first portion of class (15-20 minutes) to fill in the unit circle. This will be counted as a quiz. Exact values are required.

The remainder of class, students will be guided through the notes sheet. It introduces the students to important vocabulary that will be used in greater depth in the following class. They will learn to define periodic function, cycle, period, and amplitude.

Students will then look at the three basic trigonometric functions and calculate amplitude and period for these three. Students should notice that calculating amplitude does not work for tangent because there is no minimum or maximum. Tangent has no “amplitude” in the technical sense, but next class, they will discuss how a variable, \( a \), might change the tangent function in the case of \( y = a \tan \theta \).
Name: ______________________

Unit Circle Quiz

Directions: Name the exact coordinates in the space provided for each angle around the unit circle.

Which trig function is represented by the x-coordinate of all the points written above?

__________________________________

Which trig function is represented by the y-coordinate of all the points written above?

__________________________________

Which trig function is represented by the x-coordinate of all the points written above?

__________________________________

Which trig function is represented by the y-coordinate of all the points written above?

__________________________________
Periodic Functions

Sine, Cosine, and Tangent are all examples of Periodic Functions because the function repeats itself after a specified amount of time. One completion of the pattern is called a cycle. The period of a function is the horizontal length of one cycle. Look at the images below and calculate the period of all periodic functions. If not periodic, write not periodic.

a. 

![Image](Bellman et. al., 2005)

b. 

![Image](Bellman et. al., 2005)

c. 

![Image](Bellman et. al., 2005)

d. 

![Image](Bellman et. al., 2005)

The amplitude of a periodic function is half the difference of the maximum and minimum values. It measures the variation of a function.

![Image](Bellman et. al., 2005)

Go back up to problems (a-d) and identify the amplitude of all the periodic functions.
Let’s take a look at our three basic trig functions:

a. \( y = \sin \theta \)

Period: 

Amplitude: 

(b) \( y = \cos \theta \)

Period: 

Amplitude: 

(c) \( y = \tan \theta \)

Period: 

Amplitude:
Day 8 & 9- Part 2 of Trig Transformations

Objectives: Students will discover transformations upon the sine, cosine, and tangent functions including changes in amplitude, frequency, and vertical shift.


Materials:

- Graphing Calculators (such as TI-83+, or TI-84)
- Student note sheets
- Display copy of notes
- Colored pencils

Lesson Procedure:

Students will use their calculators in order to view a lot of different trig functions in a short period of time. Even so, it is expected that the included notes pages will take over 1 class period to complete. Therefore it has been allotted two class periods. If on day 2, the notes are finished early, it is possible to begin day 10 of the unit as day 10 is the last day of the unit before review.

After the teacher verifies all students know how to change their calculator to be in degree mode, and helps students set their graphs to an appropriate window, students can complete the first two pages of notes with a partner with the teacher circulating the classroom to answer questions. The class will come back together to make sure all students have answered the question at the bottom of page 2:

**In general, explain the affect of A on the equations** \( y = A \sin(B\theta) + C \), \( y = A \cos(B\theta) + C \) and \( y = A \tan(B\theta) + C \). **You may wish to use the new vocabulary word “amplitude” that we learned last class.**

B in the equations, \( y = A \sin(B\theta) + C \), \( y = A \cos(B\theta) + C \) and \( y = A \tan(B\theta) + C \) may require some more guidance by the teacher (especially to understand the differences between period and frequency). Therefore, it is suggested to do these parts together. When graphing multiple graphs in the same window, suggest that students use a different colored pencil for each graph.

Do one vertical shift problem together as a class before having the students work on the remainder of the questions independently.
Today we will be graphing trigonometric functions of the following form:

\[ y = A \sin(B \theta) + C \]
\[ y = A \cos(B \theta) + C \]
\[ y = A \tan(B \theta) + C \]

Your goal today is to figure out what effect A, B, and C have on their parent functions, \( y = \sin \theta \), \( y = \cos \theta \), and \( y = \tan \theta \).

A: On your calculator, graph \( y = \sin \theta \). Then, ON THE SAME GRAPH, graph the functions below one at a time, and describe the transformation you see on \( y = \sin \theta \).

<table>
<thead>
<tr>
<th>( A )</th>
<th>( y = A \sin \theta )</th>
<th>Describe the transformation on ( y = \sin \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( y = 2 \sin \theta )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( y = 3 \sin \theta )</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>( y = \frac{1}{2} \sin \theta )</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>( y = -2 \sin \theta )</td>
<td></td>
</tr>
</tbody>
</table>
Try the same for cosine. On your calculator, graph \( y = \cos \theta \). Then, ON THE SAME GRAPH, graph the functions below one at a time, and describe the transformation you see on \( y = \cos \theta \).

<table>
<thead>
<tr>
<th>( A )</th>
<th>( y = A \cos \theta )</th>
<th>Describe the transformation on ( y = \cos \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( y = 2 \cos \theta )</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>( y = \frac{1}{2} \cos \theta )</td>
<td></td>
</tr>
<tr>
<td>( -2 )</td>
<td>( y = -2 \cos \theta )</td>
<td></td>
</tr>
</tbody>
</table>

Now do the same for tangent. On your calculator, graph \( y = \tan \theta \). Then, ON THE SAME GRAPH, graph the functions below one at a time, and describe the transformation you see on \( y = \tan \theta \).

<table>
<thead>
<tr>
<th>( A )</th>
<th>( y = A \tan \theta )</th>
<th>Describe the transformation on ( y = \tan \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( y = 2 \tan \theta )</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>( y = \frac{1}{2} \tan \theta )</td>
<td></td>
</tr>
<tr>
<td>( -2 )</td>
<td>( y = -2 \tan \theta )</td>
<td></td>
</tr>
</tbody>
</table>

In general, explain the affect of \( A \) on the equations \( y = A \sin(B\theta) + C \), \( y = A \cos(B\theta) + C \) and \( y = A \tan(B\theta) + C \). You may wish to use the new vocabulary word “amplitude” that we learned last class.

______________________________________________________________________________
______________________________________________________________________________
________________________

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Let’s move on to examine the $B$ in our equations.

$$y = \sin B\theta$$

On the axes below, use your graphing calculator to help draw a sketch of all three functions in the same window.

- $y = \sin \theta$
- $y = \sin 2\theta$
- $y = \sin \frac{1}{2}\theta$

The $B$ in $y = \sin B\theta$ changes the _______________ of the function.

We call $B$ the **frequency** of the function where $frequency = \frac{2\pi}{period}$.

The frequency tells us how many times the function will cycle from 0 to $2\pi$.

**Questions:**

1. Calculate the frequency and period of the graph $y = \sin \theta$.

2. Calculate the frequency and period of the graph $y = \sin 2\theta$

3. Calculate the frequency and period of the graph $y = \sin \frac{1}{2}\theta$
$y = \cos B\theta$

On the axes below, use your graphing calculator to help draw a sketch of all three functions in the same window.

- $y = \cos \theta$
- $y = \cos 4\theta$
- $y = \cos \frac{1}{4}\theta$

The $B$ in $y = \cos B\theta$ changes the ______________ of the function.

We call $B$ the **frequency** of the function where $frequency = \frac{2\pi}{period}$.

The frequency tells us how many times the function will cycle from 0 to $2\pi$.

**Questions:**

1. Calculate the frequency and period of the graph $y = \cos \theta$.

2. Calculate the frequency and period of the graph $y = \cos 4\theta$

3. Calculate the frequency and period of the graph $y = \cos \frac{1}{4}\theta$
\[ y = \tan B\theta \]

On the axes below, use your graphing calculator to help draw a sketch of all three functions in the same window. (sketch just two of these in the space provided)

- \( y = \tan \theta \)
- \( y = \tan 2\theta \)
- \( y = \tan \frac{1}{2}\theta \)

The \( B \) in \( y = \tan B\theta \) changes the _________________ of the function.

We call \( B \) the **frequency** of the function where \( \text{frequency} = \frac{\pi}{\text{period}} \). (NOTE: This formula is different for tangent. Why do you think this is?)

**Questions:**

1. Calculate the frequency and period of the graph \( y = \tan \theta \).

2. Calculate the frequency and period of the graph \( y = \tan 2\theta \)

3. Calculate the frequency and period of the graph \( y = \tan \frac{1}{2}\theta \)
Finally, we will examine what $C$ does to our equations.

1. On your calculator, graph $y = \sin(\theta)$ and $y = \sin(\theta) + 4$ in the same window. Be sure you close your parentheses on the sine function! **Record your observations below. (It may help to draw a sketch)**

2. On your calculator, graph $y = \cos(\theta)$ and $y = \cos(\theta) - 1$ in the same window. Be sure you close your parentheses on the sine function! **Record your observations below. (It may help to draw a sketch)**

3. On your calculator, graph $y = \tan(\theta)$ and $y = \tan(\theta) + 2$ in the same window. Be sure you close your parentheses on the sine function! **Record your observations below. (It may help to draw a sketch)**

4. Generalize what you saw above and draw a conclusion about what $C$ does in the equations

\[
\begin{align*}
y &= A \sin(B\theta) + C \\
y &= A \cos(B\theta) + C \\
y &= A \tan(B\theta) + C
\end{align*}
\]
Summary:

\[ y = A \sin(B\theta) + C \]

- \( |A| \): Amplitude (if \( A \) is negative, the graph gets flipped over the x-axis).
- \( B \): Frequency (How many times we see a cycle from 0 to \( 2\pi \)).
  
  \[ \text{Period} = \frac{2\pi}{\text{frequency}} \]

- \( C \): The vertical shift of the graph

\[ y = A \cos(B\theta) + C \]

- \( |A| \): Amplitude (if \( A \) is negative, the graph gets flipped over the x-axis).
- \( B \): Frequency (How many times we see a cycle from 0 to \( 2\pi \)).
  
  \[ \text{Period} = \frac{2\pi}{\text{frequency}} \]

- \( C \): The vertical shift of the graph

\[ y = A \tan(B\theta) + C \]

- \( |A| \): The vertical stretch of the graph (if \( A \) is negative, the graph gets flipped over the x-axis).
- \( B \): Frequency (How many times we see a cycle from \( \frac{\pi}{2} \) to \( \frac{\pi}{2} \)).
  
  \[ \text{Period} = \frac{\pi}{\text{frequency}} \]

- \( C \): The vertical shift of the graph
Day 10- Trigonometric Graphs and Applications

Objectives: Students will be able to look at a sine, cosine, or tangent graph and identify an equation for the graph from the picture. Students will be able to graph a sine cosine or tangent graph given an equation. Students will apply their knowledge of sine and cosine graphs to create a real world model of periodic phenomena.


Materials:
- Student note sheets
- Calculators
- Display copy of notes

(Graphs and application problems with periodic functions from the notes come from the textbook Prentice Hall Mathematics: Algebra 2 by Allan E. Bellman, Sadie Chavis Bragg, Randall I. Charles, Basia Hall, William G. Handlin, Sr., and Dan Kennedy)

Lesson Procedure:

Depending on the timing through the combined days 8 and 9, this lesson will either be started at the beginning of a new day of class, or with the time remaining at the end of day 9. It is a continuation of applying their knowledge of transformations on trigonometric graphs.

The teacher will guide students through the notes sheet, paying particular attention when students reach the application problems to make sure students gain a clear understanding how to connect data with trigonometric graphs.
Sine, Cosine, and Tangent Graphs

Using the pictures below, identify the period, amplitude, frequency, vertical shift, and write an equation for the graph shown.

1.

![Graph 1]

Amplitude: ________
Frequency: ________
Period: ________
Vertical Shift: ________
Equation: ______________________

2.

![Graph 2]

Amplitude: ________
Frequency: ________
Period: ________
Vertical Shift: ________
Equation: ______________________

3.

![Graph 3]

Amplitude: ________
Frequency: ________
Period: ________
Vertical Shift: ________
Equation: ______________________
Use the given equations to identify amplitude, period, frequency, and vertical shift, then sketch a graph of the function on the coordinate plane provided.

1. \( y = -2 \sin \frac{1}{2} \theta + 1 \)

   Amplitude:

   Frequency:

   Period:

   Vertical Shift:

2. \( y = \tan 2 \theta - 2 \)

   Frequency:

   Period:

   Vertical Shift:
Applications of periodic graphs

The table below shows the times for high tide and low tide. The markings on the side of a local pier showed a high tide of 7 ft. and a low tide of 4 ft.

<table>
<thead>
<tr>
<th>Tide Table</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>High Tide</td>
<td>4:03 A.M.</td>
</tr>
<tr>
<td>Low Tide</td>
<td>10:14 A.M.</td>
</tr>
<tr>
<td>High Tide</td>
<td>4:25 P.M.</td>
</tr>
<tr>
<td>Low Tide</td>
<td>10:36 P.M.</td>
</tr>
</tbody>
</table>

a. What is the average depth of water at the pier? What is the amplitude of the variation from the average depth?

b. How long is one cycle of the tide?

c. Write a cosine function that models the relationship between the depth of water and the time of day. Use y = 0 to represent the average depth of water. Use t = 0 to represent the time 4:03 A.M.

d. Suppose your boat needs at least 5 ft of water to approach or leave the pier. Between the hours of 4:03 A.M. and 4:25 P.M., at what times could you come and go?
Sunrise and sunset are defined as the times when someone at sea level sees the uppermost edge of the sun on the horizon. In Houston, Texas, at the spring equinox (March 21), there are 12 hours and 9 minutes of sunlight. Throughout the year, the variation from 12 hours 9 minutes of sunlight can be modeled by a sine function. The longest day (June 21) has 1 hour 55 minutes more sunlight than at equinox. The shortest day (December 21) has 1 hour 55 minutes less sunlight.

a. Define the independent and dependent variables for a function that models the variation in hours of sunlight in Houston.

b. What are the amplitude and period of the function measured in days?

c. Write a function that relates the number of days away from the spring equinox to the variation in hours of sunlight in Houston.

d. Use your function from part (c). In Houston, about how much less sunlight does February 14 have than March 21?
**Day 11- Review**

**Objectives:** Students will review all of the learning standards in this unit in preparation for the unit test

**Learning Standards (CCSS):**
- CCSS.Math.Content.HSF-TF.A.1
- CCSS.Math.Content.HSF-TF.A.2
- CCSS.Math.Content.HSF-TF.A.3
- CCSS.Math.Content.HSF-TF.B.5

**Materials:**
- Student review sheets
- Calculators

(Graphs with periodic functions from the notes come from the textbook *Prentice Hall Mathematics: Algebra 2* by Allan E. Bellman, Sadie Chavis Bragg, Randall I. Charles, Basia Hall, William G. Handlin, Sr., and Dan Kennedy)

**Lesson Procedure:**

The teacher may choose to implement this review in whichever way they see fit. Two different options are described below, or the teacher may choose to use their own method:

1. Students receive review packet on day 10 to do for homework. Students work in pairs and are assigned a problem (or several problems) to become “experts” on. On review day, students will present their solutions to their “expert” problems to the whole class.

2. Students are given the packet on review day and are assigned small groups to work on the problems together. They may choose to skip around the packet to areas they feel they need the most review. When a student is struggling with a problem, he is to ask the group to help understand. If the whole group does not know, they will raise their hands to get the teacher’s assistance. The teacher will move around the classroom and help groups out as necessary.
Name:____________________
Unit Review

NOTE: Make sure you are completing ALL PARTS of each problem!

1. Convert the following angles in degree measure to radians. Leave your answer in terms of $\pi$.

   a. 72°  
   b. 134°  
   c. 315°

2. Convert the following angles in radian measure to degrees. Round your answer to the nearest tenth of a degree.

   a. $\frac{7\pi}{3}$  
   b. $-\frac{4\pi}{7}$  
   c. $4\pi$

3. Sketch the angle $\frac{13\pi}{15}$ in the coordinate plane below. The terminal side is in quadrant ______.
4. State the **exact** numerical value of the given expressions.

   a. \( \cos 3\pi \)  
   c. \( \tan \frac{2\pi}{3} \)

   b. \( \sin \left( -\frac{\pi}{3} \right) \)  
   d. \( -\sin \left( -\frac{5\pi}{4} \right) \)

5. Find all values of \( \theta \) on the interval \( 0 \leq \theta \leq 2\pi \) for which \( \cos \theta = \cos 2\theta \). Use of the graph is optional.
6. State the amplitude, frequency, period, vertical shift, and write an equation for the sine curve shown below.

![Sine Curve](image1)

| Amplitude: |
| Frequency: |
| Period:    |
| Vertical Shift: |
| Equation:  |

(Bellman et. al., 2005)

7. The graphing calculator screen shown below displays the interval 0 to $2\pi$. What is the period of this graph?

![Graphing Calculator](image2)

(Bellman et. al., 2005)

8. Find the amplitude and period of the cosine function shown in the calculator window below. Write an equation for this function.

![Cosine Function](image3)

(Bellman et. al., 2005)
9. Write a possible equation for the function shown below.

(Bellman et. al., 2005)

10. Consider the function \( y = 2 \cos\left(\frac{1}{2} \theta\right) + 1 \)

   a. State the domain and range of the above function

   b. Sketch a graph of the function in the axes below
Day 12- Test

Objectives: Students will complete a summative assessment on the entire unit of material.

Learning Standards (CCSS): CCSS.Math.Content.HSF-TF.A.1
- CCSS.Math.Content.HSF-TF.A.2
- CCSS.Math.Content.HSF-TF.A.3
- CCSS.Math.Content.HSF-TF.B.5

Materials:
- Unit tests
- Calculators


Lesson Procedure: Students will use the entire period to take the unit test.)
Directions: Please read each question carefully and make sure you have answered all parts. Show all work.

1. For each of the two functions below, state whether or not the function is periodic. If it is periodic, state the period. If it is not, explain why.

   a. 
   
   ![Graph](image1)
   
   (Bellman et. al., 2005)

   b. 
   
   ![Graph](image2)
   
   (Bellman et. al., 2005)

2. Convert the following angles in radian measure to degrees. Round your answer to the nearest tenth of a degree.

   a. \(-\frac{2\pi}{5}\) 
   
   b. \(\frac{7\pi}{9}\) 
   
   c. \(\frac{11\pi}{13}\)
3. Convert the following angles in degree measure to radians. Leave your answer in terms of π.
   a. 18°
   b. 202°
   c. 225°

4. State the exact numerical value of the given expressions.
   a. \( \sin \frac{5\pi}{3} \)
   b. \( -\sin \left( \frac{5\pi}{4} \right) \)
   c. \( \tan \frac{\pi}{2} \)
   d. \( \cos \left( -\frac{2\pi}{3} \right) \)

5. Sketch the angle \( -\frac{7\pi}{6} \) in the coordinate plane to the right.
   The terminal side is in quadrant _____.

6. Write the equation of a function that is a transformation of \( y = \sin \theta \) so that its amplitude is 4 and its minimum value is 1. Show all your work.
7. Find the period and amplitude of the cosine function shown below then state a possible equation for the given function.

8. Write a sine function for the given description. Assume that \( a > 0 \).
   Amplitude: \( \frac{\pi}{3} \); Period: 3

9. Write an equation for the tangent function shown below.
10. You are sitting on a pier watching the waves when you notice a bottle in the water. The bottle bobs so that it is between 2.5 ft and 4.5 ft below the pier. You know that you can reach 3 ft below the pier. Suppose the bottle reaches its highest point every 5 s.

a. Sketch a graph of the bottle’s distance below the pier for 15 s. Assume that at $t = 0$, the bottle is closest to the pier.

b. Find the period and amplitude of the function.

c. Use your graph to estimate the length of time the bottle is within reach during each cycle.
Chapter 4: Reflection

At the conclusion of this unit, students should be comfortable using reference triangles to calculate specified values of sine cosine and tangent using both degrees and radians. Students should have a developed understanding of these three functions in relation to the unit circle, and the way they look graphed in the Cartesian plane. Students should develop the ability to identify domain, range, amplitude and frequency for these three trigonometric functions.

There are additional aspects of trigonometry that are worthwhile to pursue, namely, and Algebra 2/Trigonometry student ought to be familiar with inverse trigonometric functions. These were not included within this introductory unit to the basic trigonometric functions and their relationship to the unit circle, but may be the next logical topic to pursue within the classroom. Though this lesson seeks to incorporate researched strategies to align itself with the CCSS, it should be noted to the reader that the unit included within has not been implemented in the classroom setting as it is written. It therefore, may be necessary for a teacher to modify the timing or implementation of these lessons to suit the needs of his or her own unique classroom situation.
References


Appendix

Answer Keys to Student Notes Pages and Unit Test

Day 1 Notes Key 71
Day 2 Notes Key 72
Day 5 Notes Key 75
Day 6 Notes Key 78
Unit Circle Quiz Key 80
Day 7 Notes Key 81
Day 8&9 Notes Key 83
Day 10 Notes Key 90
Unit Review Key 94
Unit Test Key 98
What is a Radian?

8. Take your paper plate and fold it in half. You have just found the diameter of your plate!

9. Cut a piece of string equal to the length of the diameter of your circle.

10. Measure around the outside of your plate with the piece of string.
   
   c. The outside of the circle measures just over \( \frac{3}{3} \) lengths of the piece of string.
   
   d. The formula for the circumference of a circle is \( C = \pi D \) where \( D \) is the diameter of the circle. How does this formula connect to your answer in part (a)?

   __________________________________________________________________________
   __________________________________________________________________________
   __________________________________________________________________________

11. Now fold your plate in half again to form some axes. Use your pencil to draw in these axes. Label the center O. Label a point on the axes along the rim of the circle A (see diagram).

12. Cut a piece of string to the length of the radius of the circle. Starting at A, measure along the rim of the circle as far as the string will go. Label the other endpoint of the string B, and draw in angle \( \angle AOB \) (see diagram).

   *** The measure of the central angle \( \angle AOB \) whose arc length is 1 radius is one radian.***

13. Estimate the number of degrees in one radian. **Record your guess. Answers vary**
   
   Now use a protractor to measure the angle. **Record the measured value. Approxx. 57°**

14. Now estimate the number of radians in a semicircle. Starting at point B, use the radius string to continue to measure out angles of 1 radian. **Number of radians in a semicircle: Just over 3**

   Conclusion: There are ___ \( \pi \) radians in a semicircle and ___ \( 2\pi \) radians in a circle.
Degrees $\leftrightarrow$ Radians

Last class, we learned that there are $\frac{\pi}{2}$ radians in a semicircle, and a semicircle is $180^\circ$ degrees.

Use this knowledge to set up a proportion. \(\frac{\text{Degrees}}{\text{Radians}}\)

\[
\frac{\text{degrees}}{\text{radians}} = \frac{180}{\pi}
\]

Because degrees and radians are proportional, we can use this ratio and set up a proportion. Do this to convert $30^\circ$ into radians.

\[
\frac{180}{\pi} = \frac{30}{x}
\]

\[
180x = 30\pi
\]

\[
x = \frac{30\pi}{180} = \frac{\pi}{6}
\]

\[
30^\circ = \frac{\pi}{6} \text{ radians}
\]

Use a proportion to convert $\frac{4\pi}{3}$ radians into degrees.

\[
\frac{180}{\pi} = \frac{x}{\frac{4\pi}{3}}
\]

\[
x = 240
\]

\[
\frac{4\pi}{3} \text{ radians} = 240^\circ
\]

Can you think of an easier way to convert radians to degrees? Try it with the same problem above.

Substitute $\pi = 180^\circ$

\[
\frac{4\pi}{3} = \frac{4(180)}{3} = 240^\circ
\]
Now try these on your own! Convert from degrees to radians. Reduce fractions to simplest form.

\[ \begin{align*}
e. \quad -32^\circ &= -\frac{8\pi}{45} \\
g. \quad 83^\circ &= \frac{83\pi}{180}
\end{align*} \]

\[ \begin{align*}
f. \quad 215^\circ &= \frac{43\pi}{36} \\
h. \quad 400^\circ &= \frac{20\pi}{9}
\end{align*} \]

Convert the following radian measures back to degrees.

\[ \begin{align*}
e. \quad \frac{\pi}{9} &= 20^\circ \\
g. \quad 7\pi &= 1260^\circ
\end{align*} \]

\[ \begin{align*}
f. \quad \frac{3\pi}{4} &= 135^\circ \\
h. \quad -\frac{5\pi}{4} &= -225^\circ
\end{align*} \]

Usually, we don’t put any units on radial measures. What’s one way we might be able to tell if an angle is measured in radians?

No ° symbol is written, usually radian measures have a π in them (but not always)
Recall: Each quadrant of the coordinate plane is given a number. We label these I, II, III, and IV. They are labeled in a counterclockwise direction. Label the set of axes below.

Definitions:

Initial side of an angle - the side that the measurement of an angle starts from

Terminal side of an angle - the side at which the measurement of an angle ends

Normally we place the initial side along the positive x axis and call this 0°. We measure angles in a counterclockwise direction.

Draw each of the following angles into the axes above

- Label the positive x-axis 0 for zero degrees or radians.
- Determine the radian measures for 90°, 180°, 270°, and 360°, and place these measures where they belong on the axes above.
- In RED, draw an angle whose terminal side is in quadrant III.
- In GREEN, draw the terminal side of the angle whose measure is $\frac{5\pi}{6}$. In which quadrant does this angle end? _____ II _____
- In BLUE, draw the terminal side of the angle whose measure is $-\frac{\pi}{4}$. In which quadrant does this angle end? _____ IV _____
Sine, Cosine, and Tangent

Use your Unit Circle to help fill out the following chart for sine values and cosine values of $\theta$. Don’t worry about the tangent row for now!

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>$\pi/6$</th>
<th>$\pi/4$</th>
<th>$\pi/3$</th>
<th>$\pi/2$</th>
<th>$2\pi/3$</th>
<th>$3\pi/4$</th>
<th>$5\pi/6$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{-\sqrt{2}}{2}$</td>
<td>$\frac{-\sqrt{3}}{2}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td>0</td>
<td>$\frac{\sqrt{3}}{3}$</td>
<td>1</td>
<td>$\sqrt{3}$</td>
<td>Undef.</td>
<td>$-\sqrt{3}$</td>
<td>$-1$</td>
<td>$-\frac{\sqrt{3}}{3}$</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$7\pi/6$</th>
<th>$5\pi/4$</th>
<th>$4\pi/3$</th>
<th>$3\pi/2$</th>
<th>$5\pi/3$</th>
<th>$7\pi/4$</th>
<th>$11\pi/6$</th>
<th>$2\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$</td>
<td>$\frac{-1}{2}$</td>
<td>$\frac{-\sqrt{2}}{2}$</td>
<td>$\frac{-\sqrt{3}}{2}$</td>
<td>$-1$</td>
<td>$\frac{-\sqrt{3}}{2}$</td>
<td>$\frac{-\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>$\frac{-\sqrt{3}}{2}$</td>
<td>$\frac{-\sqrt{2}}{2}$</td>
<td>$\frac{-1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td>$\frac{\sqrt{3}}{3}$</td>
<td>1</td>
<td>$\sqrt{3}$</td>
<td>Undef.</td>
<td>$-\sqrt{3}$</td>
<td>$-1$</td>
<td>$-\frac{\sqrt{3}}{3}$</td>
<td>0</td>
</tr>
</tbody>
</table>

You may recall $\tan \theta = \frac{\sin \theta}{\cos \theta}$. Use this fact to calculate the value of $\tan \theta$ for all $\theta$ above. Use the space below for any necessary calculations.
We know how sine and cosine relate to the unit circle, but what does tangent look like? Draw it in below:

Now let’s graph our three trigonometric functions in the spaces provided below:

\[ f(\theta) = \sin \theta \]

\[ f(\theta) = \cos \theta \]
f(θ) = tan θ

Questions:

1. What do you think will happen if we go beyond 2π in either direction on the x-axis? Why?

   The graphs will continue in the same cyclic pattern because points beyond 360 degrees are related to the same points around the unit circle.

2. State the domain and range of each function:

   \[ f(θ) = \sin θ \quad \text{Domain:} \ (-∞, ∞) \quad \text{Range:} \ [-1, 1] \]

   \[ f(θ) = \cos θ \quad \text{Domain:} \ (-∞, ∞) \quad \text{Range:} \ [-1, 1] \]

   \[ f(θ) = \tan θ \quad \text{Domain:} \ \mathbb{R} \text{ except } \frac{π}{2} \text{ where } x \text{ is an odd number} \quad \text{Range:} \ (-∞, ∞) \]

3. Is it possible for \( \sin θ = 5 \)? Explain why or why not.

   No. It is not in the domain of the function.

4. Is it possible for \( \tan θ = 5 \)? Explain why or why not.

   Yes. It is in the domain of the function.
Using Reference Triangles to Find Exact Trig Values

To remember where Sine, Cosine, and Tangent are positive, we can use the mnemonic ASTC (All Students Take Calculus).

- In Quadrant I, All three trig functions are positive.
- In Quadrant II, Sine is positive, the other two are negative.
- In Quadrant III, Tangent is positive, the other two are negative.
- In Quadrant IV, Cosine is positive, the other two are negative.

Using your paper plate, identify all locations where $|\sin \theta| = \frac{1}{2}$ (the absolute value of sine)

This should happen 4 times. Each time, you should notice we use a 30-60-90 right triangle in each of the four quadrants. These are Reference Triangles. You make a reference triangle by dropping a perpendicular from the terminal side of the angle to the x-axis.

**Draw a reference triangle for $\frac{7\pi}{6}$.**

![Reference Triangle Diagram]

What is the central angle in the reference triangle you drew?

$$\frac{\pi}{6} \quad (30^\circ)$$

We call the above angle the Reference Angle. Calculate the sine of this reference angle.

$$\sin \left( \frac{\pi}{6} \right) = \frac{1}{2}$$

Is sine Positive or Negative in Quadrant III?

Negative

What is $\sin \left( \frac{7\pi}{6} \right)$?

$$\sin \left( \frac{7\pi}{6} \right) = -\frac{1}{2}$$
Now use reference triangles to calculate $\cos \frac{11\pi}{4}$. Include a picture of the reference triangle in your solution. (Tip: If you are ever struggling with radian measure, you can always convert it into degrees)

\[
\cos \left( \frac{11\pi}{4} \right) = -\cos \left( \frac{\pi}{4} \right) = -\frac{\sqrt{2}}{2}
\]

Try these on your own:

a. $\tan \frac{4\pi}{3} = \sqrt{3}$

b. $\cos \frac{11\pi}{3} = \frac{1}{2}$

c. $\sin \frac{7\pi}{2} = -1$

d. $\cos \frac{13\pi}{4} = -\frac{\sqrt{2}}{2}$

**Conclusion:** Reference triangles allow us to find the exact value of multiples of the angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ by knowing trig values for these three values and some simple facts about the signs of the functions in the associated quadrant.
Name: ____________________
**Unit Circle Quiz**

**Directions:** Name the exact coordinates in the space provided for each angle around the unit circle.

Which trig function is represented by the x-coordinate of all the points written above?

__________________________

Which trig function is represented by the y-coordinate of all the points written above?

__________________________
Periodic Functions

Sine, Cosine, and Tangent are all examples of **Periodic Functions** because the function repeats itself after a specified amount of time. One completion of the pattern is called a **cycle**. The **period** of a function is the horizontal length of one cycle. Look at the images below and **calculate the period of all periodic functions**. If not periodic, write **not periodic**.

a. Not Periodic

![Graph of a non-periodic function](image)

(b) Periodic. Period = 4, Amplitude = 2

![Graph of a periodic function with period 4 and amplitude 2](image)

c. Periodic. Period = 6, Amplitude = 1.5

![Graph of a periodic function with period 6 and amplitude 1.5](image)

d. Periodic. Period = 7, Amplitude = 4.5

![Graph of a periodic function with period 7 and amplitude 4.5](image)

The **amplitude** of a periodic function is half the difference of the maximum and minimum values. It measures the variation of a function.

![Graph of a periodic function with amplitude](image)

**Go back up to problems (a-d) and identify the amplitude of all the periodic functions.**
Let’s take a look at our three basic trig functions:

a. \( y = \sin \theta \)
   
   Period: \( 2\pi \)
   
   Amplitude: 1

\( \text{(Bellman et. al., 2005)} \)

b. \( y = \cos \theta \)
   
   Period: \( 2\pi \)
   
   Amplitude: 1

\( \text{(Bellman et. al., 2005)} \)

c. \( y = \tan \theta \)
   
   Period: \( \pi \)
   
   Amplitude: Does not have an amplitude

\( \text{(Bellman et. al., 2005)} \)
Transformations on Sine, Cosine, and Tangent

NOTE: WHEN WORKING WITH TRIG ON YOUR CALCULATOR, IT IS VERY IMPORTANT THAT YOU PUT YOUR CALCULATOR IN THE CORRECT MODE! WE WILL BE WORKING WITH RADIANS, SO PUT YOUR CALCULATOR IN RADIANT MODE!

Today we will be graphing trigonometric functions of the following form:

\[ y = A \sin(B\theta) + C \]
\[ y = A \cos(B\theta) + C \]
\[ y = A \tan(B\theta) + C \]

Your goal today is to figure out what effect A, B, and C have on their parent functions, \( y = \sin \theta \), \( y = \cos \theta \), and \( y = \tan \theta \).

A: On your calculator, graph \( y = \sin \theta \). Then, ON THE SAME GRAPH, graph the functions below one at a time, and describe the transformation you see on \( y = \sin \theta \).

<table>
<thead>
<tr>
<th>( A )</th>
<th>( y = A \sin \theta )</th>
<th>Describe the transformation on ( y = \sin \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( y = 2 \sin \theta )</td>
<td>Vertical stretch (taller)- amplitude goes from 1 to 2.</td>
</tr>
<tr>
<td>3</td>
<td>( y = 3 \sin \theta )</td>
<td>Vertical stretch (taller)- amplitude goes from 1 to 3.</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>( y = \frac{1}{2} \sin \theta )</td>
<td>Vertical shrink (shorter)- amplitude goes from 1 to ( \frac{1}{2} ).</td>
</tr>
<tr>
<td>( -2 )</td>
<td>( y = -2 \sin \theta )</td>
<td>Vertical stretch (taller), and reflect over x-axis. Amplitude goes from 1 to 2.</td>
</tr>
</tbody>
</table>
Try the same for cosine. On your calculator, graph \( y = \cos \theta \). Then, ON THE SAME GRAPH, graph the functions below one at a time, and describe the transformation you see on \( y = \cos \theta \).

<table>
<thead>
<tr>
<th>( A )</th>
<th>( y = A \cos \theta )</th>
<th>Describe the transformation on ( y = \cos \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( y = 2 \cos \theta )</td>
<td>Vertical stretch (taller)- amplitude goes from 1 to 2.</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>( y = \frac{1}{2} \cos \theta )</td>
<td>Vertical shrink (shorter)- amplitude goes from 1 to ( \frac{1}{2} ).</td>
</tr>
<tr>
<td>( -2 )</td>
<td>( y = -2 \cos \theta )</td>
<td>Vertical stretch (taller), and reflect over x-axis. Amplitude goes from 1 to 2.</td>
</tr>
</tbody>
</table>

Now do the same for tangent. On your calculator, graph \( y = \tan \theta \). Then, ON THE SAME GRAPH, graph the functions below one at a time, and describe the transformation you see on \( y = \tan \theta \).

<table>
<thead>
<tr>
<th>( A )</th>
<th>( y = A \tan \theta )</th>
<th>Describe the transformation on ( y = \tan \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( y = 2 \tan \theta )</td>
<td>Vertical stretch (graph appears steeper).</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>( y = \frac{1}{2} \tan \theta )</td>
<td>Vertical shrink (graph appears less steep).</td>
</tr>
<tr>
<td>( -2 )</td>
<td>( y = -2 \tan \theta )</td>
<td>Vertical stretch and reflection on the x-axis.</td>
</tr>
</tbody>
</table>

In general, explain the affect of \( A \) on the equations \( y = A \sin(B\theta) + C \), \( y = A \cos(B\theta) + C \) and \( y = A \tan(B\theta) + C \). You may wish to use the new vocabulary word “amplitude” that we learned last class.

---

‘A’ changes the amplitude by vertically stretching or shrinking the graph.
If \( A < 0 \), the graph will be reflected on the x-axis.
Let’s move on to examine the $B$ in our equations.

$y = \sin B\theta$

On the axes below, use your graphing calculator to help draw a sketch of all three functions in the same window.

- $y = \sin \theta$
- $y = \sin 2\theta$
- $y = \sin \frac{1}{2} \theta$

The $B$ in $y = \sin B\theta$ changes the __________ of the function.

We call $B$ the **frequency** of the function where $\text{frequency} = \frac{2\pi}{\text{period}}$.

The frequency tells us how many times the function will cycle from 0 to $2\pi$.

**Questions:**

1. Calculate the frequency and period of the graph $y = \sin \theta$.
   
   Frequency = 1
   
   Period = $2\pi$

2. Calculate the frequency and period of the graph $y = \sin 2\theta$
   
   Frequency = 2
   
   Period = $\pi$

3. Calculate the frequency and period of the graph $y = \sin \frac{1}{2} \theta$
   
   Frequency = $\frac{1}{2}$
   
   Period = $4\pi$
\[ y = \cos B \theta \]

On the axes below, use your graphing calculator to help draw a sketch of all three functions in the same window.

- \( y = \cos \theta \)
- \( y = \cos 4\theta \)
- \( y = \cos \frac{1}{4} \theta \)

The \( B \) in \( y = \cos B \theta \) changes the _______ of the function.

We call \( B \) the **frequency** of the function where \( \text{frequency} = \frac{2\pi}{\text{period}} \).

The frequency tells us how many times the function will cycle from 0 to \( 2\pi \).

**Questions:**

1. Calculate the frequency and period of the graph \( y = \cos \theta \).
   
   Frequency = 1
   
   Period = \( 2\pi \)

2. Calculate the frequency and period of the graph \( y = \cos 4\theta \)
   
   Frequency = 4
   
   Period = \( \frac{\pi}{2} \)

3. Calculate the frequency and period of the graph \( y = \cos \frac{1}{4} \theta \)
   
   Frequency = \( \frac{1}{4} \)
   
   Period = \( 8\pi \)
\[ y = \tan B\theta \]

On the axes below, use your graphing calculator to help draw a sketch of all three functions in the same window. (sketch just two of these in the space provided)

- \( y = \tan \theta \)
- \( y = \tan 2\theta \)
- \( y = \tan \frac{1}{2}\theta \)

The \( B \) in \( y = \tan B\theta \) changes the \underline{period} of the function.

We call \( B \) the \textbf{frequency} of the function where \( frequency = \frac{\pi}{period} \) (NOTE: This formula is different for tangent. Why do you think this is?)

because the normal period for \( \tan \theta \) is \( \pi \), not \( 2\pi \) like sine and cosine.

Questions:

4. Calculate the frequency and period of the graph \( y = \tan \theta \).

   Frequency = 1
   Period = \( \pi \)

5. Calculate the frequency and period of the graph \( y = \tan 2\theta \)

   Frequency = 2
   Period = \( \frac{\pi}{2} \)

6. Calculate the frequency and period of the graph \( y = \tan \frac{1}{2}\theta \)

   Frequency = \( \frac{1}{2} \)
   Period = \( 2\pi \)
Finally, we will examine what $C$ does to our equations.

1. On your calculator, graph $y = \sin(\theta)$ and $y = \sin(\theta) + 4$ in the same window. Be sure you close your parentheses on the sine function! **Record your observations below. (It may help to draw a sketch)**

   The graph shifted up 4 units

2. On your calculator, graph $y = \cos(\theta)$ and $y = \cos(\theta) - 1$ in the same window. Be sure you close your parentheses on the sine function! **Record your observations below. (It may help to draw a sketch)**

   The graph shifted down 1 unit

3. On your calculator, graph $y = \tan(\theta)$ and $y = \tan(\theta) + 2$ in the same window. Be sure you close your parentheses on the sine function! **Record your observations below. (It may help to draw a sketch)**

   The graph shifted up 2 units

4. Generalize what you saw above and draw a conclusion about what $C$ does in the equations

   $y = A \sin(B\theta) + C$
   $y = A \cos(B\theta) + C$
   $y = A \tan(B\theta) + C$

   ‘$C$’ is a vertical shift.

   If $C > 0$, graph shifts up. If $C < 0$, graph shifts down. If $C = 0$, there is no vertical shift.
Summary:

\[ y = A \sin(B\theta) + C \]

- \(|A|\) = Amplitude (if \(A\) is negative, the graph gets flipped over the x-axis).
- \(B\) = Frequency (How many times we see a cycle from 0 to 2\(\pi\)).

\[ \text{Period} = \frac{2\pi}{\text{frequency}} \]

- \(C\) = The vertical shift of the graph

\[ y = A \cos(B\theta) + C \]

- \(|A|\) = Amplitude (if \(A\) is negative, the graph gets flipped over the x-axis).
- \(B\) = Frequency (How many times we see a cycle from 0 to 2\(\pi\)).

\[ \text{Period} = \frac{2\pi}{\text{frequency}} \]

- \(C\) = The vertical shift of the graph

\[ y = A \tan(B\theta) + C \]

- \(|A|\) = The vertical stretch of the graph (if \(A\) is negative, the graph gets flipped over the x-axis).
- \(B\) = Frequency (How many times we see a cycle from \(-\frac{\pi}{2}\) to \(\frac{\pi}{2}\)).

\[ \text{Period} = \frac{\pi}{\text{frequency}} \]

- \(C\) = The vertical shift of the graph
Sine, Cosine, and Tangent Graphs

Using the pictures below, identify the period, amplitude, frequency, vertical shift, and write an equation for the graph shown.

1. 

   ![Graph 1](Bellman et al., 2005)

   - Amplitude: $1$
   - Frequency: $1$
   - Period: $2\pi$
   - Vertical Shift: $3$ (up)
   - Equation: $y = \sin \theta + 3$

2. 

   ![Graph 2](Bellman et al., 2005)

   - Amplitude: $3$
   - Frequency: $2$
   - Period: $\pi$
   - Vertical Shift: $0$
   - Equation: $y = -3 \cos 2\theta$

3. 

   ![Graph 3](Bellman et al., 2005)

   - Amplitude: $3$
   - Frequency: $\frac{1}{2}$
   - Period: $2\pi$
   - Vertical Shift: $0$
   - Equation: $y = \tan \frac{1}{2}\theta$
Use the given equations to identify amplitude, period, frequency, and vertical shift, then sketch a graph of the function on the coordinate plane provided.

1. \( y = -2 \sin \frac{1}{2} \theta + 1 \)

   - Amplitude: 2 (reflected over x-axis)
   - Frequency: \(\frac{1}{2}\)
   - Period: \(4\pi\)
   - Vertical Shift: 1 (up)

2. \( y = \tan 2\theta - 2 \)

   - Frequency: 2
   - Period: \(\frac{\pi}{2}\)
   - Vertical Shift: 2 (down)
Applications of periodic graphs

The table below shows the times for high tide and low tide. The markings on the side of a local pier showed a high tide of 7 ft. and a low tide of 4 ft.

<table>
<thead>
<tr>
<th>Tide Table</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>High Tide</td>
<td>4:03 A.M.</td>
</tr>
<tr>
<td>Low Tide</td>
<td>10:14 A.M.</td>
</tr>
<tr>
<td>High Tide</td>
<td>4:25 P.M.</td>
</tr>
<tr>
<td>Low Tide</td>
<td>10:36 P.M.</td>
</tr>
</tbody>
</table>

a. What is the average depth of water at the pier? What is the amplitude of the variation from the average depth?

Average depth is 5.5 ft

Amplitude of variation is 1.5 ft

b. How long is one cycle of the tide?

742 minutes (12 hours 22 minutes)

c. Write a cosine function that models the relationship between the depth of water and the time of day. Use $y = 0$ to represent the average depth of water. Use $t = 0$ to represent the time 4:03 A.M.

$$y = 1.5 \cos \frac{360}{742} t$$

d. Suppose your boat needs at least 5 ft of water to approach or leave the pier. Between the hours of 4:03 A.M. and 4:25 P.M., at what times could you come and go?

Pier accessible from 4:03 A.M. to 7:49 A.M.

And from 12:39 P.M. to 4:25 P.M.
Sunrise and sunset are defined as the times when someone at sea level sees the uppermost edge of the sun on the horizon. In Houston, Texas, at the spring equinox (March 21), there are 12 hours and 9 minutes of sunlight. Throughout the year, the variation from 12 hours 9 minutes of sunlight can be modeled by a sine function. The longest day (June 21) has 1 hour 55 minutes more sunlight than at equinox. The shortest day (December 21) has 1 hour 55 minutes less sunlight.

a. Define the independent and dependent variables for a function that models the variation in hours of sunlight in Houston.
   - Independent: Day of the year (where $t=0$ represents March 21)
   - Dependent: Hours of sunlight (where $y=0$ represents the hours of sunlight on March 21)

b. What are the amplitude and period of the function measured in days?
   - Amplitude = 115 minutes
   - Period = 365 days

c. Write a function that relates the number of days away from the spring equinox to the variation in hours of sunlight in Houston.
   \[ y = 115 \sin \frac{360}{365} t \]

d. Use your function from part (c). In Houston, about how much less sunlight does February 14 have than March 21?
   - 1 hour 5 minutes less sunlight on February 14 than March 21
Name: ____________________

Unit Review

NOTE: Make sure you are completing ALL PARTS of each problem!

1. Convert the following angles in degree measure to radians. Leave your answer in terms of $\pi$.

   a. $72^\circ \quad \frac{2\pi}{5}$
   
   b. $134^\circ \quad \frac{67\pi}{90}$
   
   c. $315^\circ \quad \frac{7\pi}{4}$

2. Convert the following angles in radian measure to degrees. Round your answer to the nearest tenth of a degree.

   a. $\frac{7\pi}{3} \quad 420^\circ$
   
   b. $-\frac{4\pi}{7} \quad -102.9^\circ$
   
   c. $4\pi \quad 720^\circ$

3. Sketch the angle $\frac{13\pi}{15}$ in the coordinate plane below. The terminal side is in quadrant II.
4. State the **exact** numerical value of the given expressions.

   a. \( \cos 3\pi = -1 \)  
   b. \( \sin \left( -\frac{\pi}{3} \right) = -\frac{\sqrt{3}}{2} \)  
   c. \( \tan \frac{2\pi}{3} = -\sqrt{3} \)  
   d. \( -\sin \left( -\frac{5\pi}{4} \right) = \frac{\sqrt{2}}{2} \)

5. Find all values of \( \theta \) on the interval \( 0 \leq \theta \leq 2\pi \) for which \( \cos \theta = \cos 2\theta \). Use of the graph is optional.

\[
0, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ and } 2\pi
\]
6. State the amplitude, frequency, period, vertical shift, and write an equation for the sine curve shown below.

   ![Sine Curve Image](Bellman et. al., 2005)

   Amplitude: 2.5  
   Frequency: 2  
   Period: \( \pi \)  
   Vertical Shift: 0  
   Equation: \( y = 2.5 \sin 2\theta \)

7. The graphing calculator screen shown below displays the interval 0 to 2\( \pi \). What is the period of this graph?

   ![Graphing Calculator Image](Bellman et. al., 2005)

   Period = \( \frac{\pi}{2} \)

8. Find the amplitude and period of the cosine function shown in the calculator window below. Write an equation for this function.

   ![Calculator Window Image](Bellman et. al., 2005)

   Amplitude = 1  
   Frequency= 3  
   Period = \( \frac{2\pi}{3} \)  
   Equation: \( y = \cos 3\theta \)
9. Write a possible equation for the function shown below.

\[ y = -\tan \theta \]

(Bellman et. al., 2005)

10. Consider the function \( y = 2 \cos \left( \frac{1}{2} \theta \right) + 1 \)

c. State the domain and range of the above function

   Domain: \( \mathbb{R} \)

   Range: \(-1 \leq y \leq 3\)

d. Sketch a graph of the function in the axes below
Directions: Please read each question carefully and make sure you have answered all parts. Show all work.

1. For each of the two functions below, state whether or not the function is periodic. If it is periodic, state the period. If it is not, explain why.

   a. [Graph]

      Not periodic. The graph is increasing over time and does not come back down and repeat.

   b. [Graph]

      Periodic. Period = 10

2. Convert the following angles in radian measure to degrees. Round your answer to the nearest tenth of a degree.

   a. \(-\frac{2\pi}{5}\) \(-72^\circ\)  
   b. \(\frac{7\pi}{9}\) \(140^\circ\)  
   c. \(\frac{11\pi}{13}\) \(152.3^\circ\)
3. Convert the following angles in degree measure to radians. Leave your answer in terms of \( \pi \).
   a. \( 18^\circ \) \( \frac{\pi}{10} \)
   b. \( 202^\circ \) \( \frac{101\pi}{90} \)
   c. \( 225^\circ \) \( \frac{5\pi}{4} \)

4. State the exact numerical value of the given expressions.
   a. \( \sin \frac{5\pi}{3} = \frac{-\sqrt{3}}{2} \)
   b. \( -\sin \left( \frac{5\pi}{4} \right) = \frac{-\sqrt{2}}{2} \)
   c. \( \tan \frac{\pi}{2} = \text{undefined} \)
   d. \( \cos \left( -\frac{2\pi}{3} \right) = \frac{1}{2} \)

5. Sketch the angle \( -\frac{7\pi}{6} \) in the coordinate plane to the right.
   The terminal side is in quadrant II.

6. Write the equation of a function that is a transformation of \( y = \sin \theta \) so that its amplitude is 4 and its minimum value is 1. Show all your work.

\[
y = 4 \sin \theta + 5, \text{ (optionally, students may vary the b term in } y = a \sin b \theta + c \text{ for full credit)}
\]
7. Find the period and amplitude of the cosine function shown below then state a possible equation for the given function.

\[ y = \cos 2\theta \]

8. Write a sine function for the given description. Assume that \( a > 0 \).

Amplitude: \( \frac{\pi}{3} \); Period: 3

\[ y = \frac{\pi}{3} \sin \left( \frac{2\pi}{3} \theta \right) \]

9. Write an equation for the tangent function shown below.

\[ y = \tan \frac{1}{2} \theta \]
10. You are sitting on a pier watching the waves when you notice a bottle in the water. The bottle bobs so that it is between 2.5 ft and 4.5 ft below the pier. You know that you can reach 3 ft below the pier. Suppose the bottle reaches its highest point every 5 s.

a. Sketch a graph of the bottle’s distance below the pier for 15 s. Assume that at \( t = 0 \), the bottle is closest to the pier.

```
Sketches vary. Approximately should resemble \( y = \cos \left( \frac{360}{5} \theta \right) - 3.5 \)
```

b. Find the period and amplitude of the function.

- Period = 5 seconds
- Amplitude = 1

C. Use your graph to estimate the length of time the bottle is within reach during each cycle.

Answers will vary based on accuracy of sketch drawn above.
Actual answer is 1.6 seconds