A Curriculum Project on Exponential Growth Aligned with Common Core Standards

Alyssa D. Ruscio
The College at Brockport, arusc1@u.brockport.edu

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A Curriculum Project on Exponential Growth Aligned with Common Core Standards

by

Alyssa Ruscio

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Chapter One: Introduction

In the world of education, teacher expectations and educational policies are continually changing at the state and national levels. For this reason, it is essential that teachers have access to appropriate materials which help them adapt their teaching practices so that students can have meaningful learning opportunities based on the current mandates. Since 2010, many states within the United States (US) have been in a mathematical curricular paradigm shift from the National Council of Teachers of Mathematics (NCTM) Standards to the Common Core State Standards (CCSS).

Porter, A., McMaken, J., Hwang, J., & Yang, R. (2011) compares the Common Core standards with current state standards and assessments. The CCSS represent considerable change from what states currently call for in their standards and in what they assess. Since these standards are different from current practices, teachers must adapt their instructional materials to align with the new eight mathematical practices, which are:

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning
In addition to meeting these mathematical practices, teachers need to engage students in meaningful learning through challenging application problems. Classroom teachers need to move away from a focus on worksheets, drill and memorize activities and elaborate test – coaching programs, and move toward an engaging, challenging curriculum that supports content acquisition through a range of instructional methods and practices that develop student cognitive strategies. Students are now expected to be proficient in a range of key cognitive strategies. The key cognitive strategies that CCSS include are:

1. Problem formulation – formulate a problem before leaping directly to a solution allows learners to generate hypotheses, reflect and make them aware of strategies they need to employ to solve the problem
2. Research – students should gather data or information while identifying relevant research
3. Interpretation – students need techniques for interpreting information they have gathered; within this students will build on the skill of determining relevance
4. Communication – construct an argument or presentation that derives directly from collected, analyzed and organized information
5. Precision and Accuracy – through each step in the learning process, students must exercise precision and accuracy consistent with the rules of mathematics (Conley, 2011, p.19).

The goal of this master’s thesis curriculum project is to provide a high quality Algebra 1: Exponential Functions Unit aligned to the Common Core State Standards (CCSS) that allows students to investigate new ideas through collaboration, collection of data, and exploration. The
purpose of this project is to provide Algebra I teachers with a clear, developed unit plan in which they can use in their own classroom.
Chapter Two: Literature Review

Paradigm Shift to Common Core State Standards

The Trends in International Mathematics and Science Study (TIMSS) provided reliable and timely data on the mathematics achievement of United States (US) students compared to that of student performance data from other countries. In the past, US students have been reported as scoring in the average range on the Mathematics Achievement tests of Fourth- and Eighth-Graders. The TIMSS study concluded that US performance in mathematics was declining. One hypothesis why students from other countries out-performed US students was that other nations had national standards. Before 2010 the US did not have national standards because the Constitution mandated that education was a State’s right. Another hypothesis was that the US had a broad science and mathematics curriculum. TIMSS and other international assessments determined that mathematics education in the US exposed students to a wide variation of content but rarely provided in depth focus on mathematics content. The TIMSS revealed that high-performing countries had a design principle for mathematics education that provided a deep focus on a few topics with comprehensible progressions between topics (Alberti, 2013, p.26).

For these reasons, the CCSS were designed with the intention to not only deliver on the promise of national standards, but also assure standards addressed the problem of a curriculum that was ‘a mile wide and an inch deep’ (Porter, McMaken, Hwang, & Yang, 2011, p.103). Today, forty-three states, the District of Columbia, four territories, and the Department of Defense Education Activity (DoDEA) have adopted the CCSS (CCSSI, 2012). EngageNY is a resource launched by the New York State Education Department. It is designed to help parents and educators understand and handle the new standards set forth by recently implemented changes in the education system. Aside from the Modules provided by EngageNY, there is
limited curriculum available for teachers to use to successfully implement the new CCSS standards; there are even fewer materials that implement the standards that support meaningful learning. Meaningful learning is that in which the learner chooses conscientiously to integrate new knowledge to knowledge that the learner already possesses (Novak, 2002). This implies that learning should be a connected in an ongoing building process. For this unit, students will use the skills they mastered from the Linear Functions Unit in order to help derive and retain concepts regarding Exponential Functions. Not only should students be able to make connections in order to ensure meaningful learning, they should be expected to explore and learn new ideas in a collaborative, student centered investigative approach.

With this new curriculum, many educators must shift their teaching styles from a traditional approach to reform teaching in order to successfully convey the new curriculum. The Common Core Curriculum focuses more on understanding and less on practicing methods (Porter, McMaken, Hwang, & Yang, 2011, p.103). This idea of practicing methods stems from traditional teaching (Boaler, 2008).

A problem with traditional teaching approaches is that thought is not required (Boaler, 2008). Teachers stand in front of a class demonstrating and providing methods for the majority of the class period each day while students copy the procedures down in their books. Students then individually work through problem sets of near-identical questions, practicing rules and methods. Students being taught in this passive approach follow and memorize steps and rules rather than learning to explore, inquire, and persevere in solving problems. These learners then find it challenging in the future to use such methods they were told in any new situations (Boaler, 2008).
Another major problem with this traditional teaching approach is that students work in silence. It is important that students are given the time to talk through methods so that they know whether or not they really understand them. Methods are probable to make sense if someone tells you what to do, but explaining these methods in your own words to someone else is the best way to know whether they are truly understood. The students who are conversing about their ideas are able to gain deeper understanding through explaining their work, and the ones listening are given greater access to understanding (Boaler, 2008).

Another issue with the silent approach is that it gives learners the wrong idea about mathematics.

According to Boaler (2008):

One of the most important parts of being mathematical is the action called reasoning. This requires explaining why something makes sense and how the different parts of a mathematical solution lead from one to another. Students who learn to reason and to justify their solutions are learning that mathematics is about making sense. Whenever students offer a solution to a math problem, they should know why the solution is appropriate and they should draw from mathematical rules and principles when they justify a solution rather than just saying that a textbook or teacher told them it was right. …Another reason that talking is so critical in mathematics classrooms is that when students discuss mathematics, they come to know that the subject is more than a collection of rules and methods set out in books – they realize the mathematics is a subject that they can have their own ideas about, a subject that invoke different perspectives and methods, and a subject that is connected through organizing concepts and themes. (p. 48-49)
Another problem in traditional mathematics is that students are often times learning without reality. “A realistic use of context is one where students are given real situations that need mathematical analysis, for which they do need to consider (rather than ignore) the variables” (Boaler, 2008, p.53). The use of comparative contexts in mathematics allows students to appreciate that they are learning an essential subject that helps make sense of the world, rather than a subject that is all about confusion and unnecessary rules.

All in all, reform curriculum requires students are being gradually introduced to a new concept they are being asked to interpret and apply the concept. To assess whether students are understanding methods, as opposed to just thinking that everything makes sense, they need to be solving complex problems. They should not just repeating procedures with different numbers. They need to be talking through and explaining different methods. The concept should be explored through an interesting context the learners can relate to. Students are encourage to discuss such concepts, exploring their meanings and when it could be used, and generally raise any questions or problems they have (Boaler, 2008).

**Aligning Curriculum to the Common Core State Standards**

The CCSS include eight mathematical practices (see page 2) that offer varieties of knowledge that mathematics educators at all levels should seek to develop in their students. These practices derive from the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. Along with the process standards, there are the strands of mathematical proficiency specified in the National Research Council’s report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding, procedural fluency and productive disposition (CCSS, p.5).
This curriculum is designed to address mathematical practices one through six, as well as the high school – modeling and high school – algebra standards for mathematical content. The focus of this unit is designed to allow students to explore and derive new ideas in a collaborative, student centered investigative approach. The unit on Exponential Functions for the Algebra I curriculum contains content that can be easily accessible by students if they have a deep and thorough understanding of Linear Functions. For this reason, this Exponential Growth unit should come subsequently the Linear Functions unit. This unit may serve as a resource for teachers to adopt and modify to fit the needs of their unique group of students. The concepts addressed in this unit are applicable and foundational for the students’ future years of mathematics.
Chapter Three: Unit Plan

This unit plan is intended to be completed by students working in groups of three to four due to the fact the questions are designed for collaboration. The teacher should act as a facilitator, walking around the room listening in on group discussions. It is the teacher’s job to ask prompting questions when students seem stuck or to guide students that may have fallen too far off from the main concept. However, it is important to let students make mistakes, as it is an essential part of the learning process. Some questions throughout the different lessons will need to be discussed and summarized as a whole class discussion. It is up to the teacher to decide what questions need this discussion based off student responses and small group discussions.

At the end of an investigation, give students a chance in their small groups to discuss the “Wrap It Up”. Once students have had a chance to think about it, it is important to complete these as a whole group discussion. The “Checking In” problems can be treated as homework. This can be treated as a formative or summative assessment.

Organization of Unit Plan

Within this unit there are multiple investigations. Within each investigation, students are expected to explore concepts collaboratively. It will take time for students to have a full and clear understanding on the content. At the end of the investigation, there is a ‘Wrap It Up’. Here is where students should be beyond the exploring, and feel comfortable and confident in summarizing what they took out from the problems within the investigation. In the “Checking In” followed by the “Wrap It Up”, students will then apply their knowledge from the previous investigation to an extension problem.
Lesson 1: Exponential Growth

Watch the following YouTube clip: https://www.youtube.com/watch?v=5ZTm-iYUpm4

In the popular book and movie *Pay It Forward*, 12-year old Trevor McKinney gets a challenging assignment from his social studies teacher. *Think of an idea for world change, and put it into practice!*

Trevor came up with an idea that fascinates his mother, his teacher, and his classmates. He suggested that he would do something really good for three people. Then when they ask how they can pay him back for the good deeds, he would tell them to “pay it forward” – each doing something good for three other people.

Trevor figured that those three people would do something good for a total of nine others. Those nine others would do something good for 27 others, and so on. He was sure that before long there would be good things happening to billions of people all around the world.

Think About It...

Continue Trevor’s kind of Pay It Forward thinking.

a) How many people would receive a Pay It Forward good deed at each of the next several stages of the process?

b) What is your best guess about the number of people who would receive Pay It Forward good deeds at the tenth stage of the process?

c) Which of the graphs do you think is most likely to represent the pattern by which the number of people receiving Pay It Forward good deeds increases as the process continues over time? Describe the pattern of change for each graph.
In this lesson, you will discover answers to questions like these and find strategies for analyzing patterns of change called *exponential growth*. You will also discover some basic properties of exponents that allow you to write exponential expressions in useful equivalent forms.

### Investigation 1

The number of good deeds in the Pay It Forward pattern can be represented by a *tree graph* that starts like this:

The vertices represent the people who receive and do good deeds. Each edge represents a good deed done by one person for another.

1. At the start of the Pay It Forward process, only one person does good deeds – for three new people. In the next stage, the three new people each do good things for three more new people. In the next stage, nine people each do good things for three more new people, and so on, with no person receiving more than one good deed.
   a. Make a table that shows the number of people who will receive good deeds at each of the next seven stages of the Pay It Forward process. Then plot the *(stage, number of good deeds)* data in your calculator.

<table>
<thead>
<tr>
<th>Stage of Process</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Good Deeds</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. How does the number of good deeds at each stage grow as the tree progresses? How is that pattern of change shown in the plot of the data?

c. How many stages of the Pay It Forward process will be needed before a total of at least 25,000 good deeds will be done?
2. Consider now how the number of good deeds would grow if each person touched by the Pay It Forward process were to do good deeds for only two other new people, instead of three.

   a. Make a tree graph for several stages of this Pay It Forward process.

   b. Make a table showing the number of good deeds increase as the Pay It Forward process progress and plot those sample (stage, number of good deeds) values.

<table>
<thead>
<tr>
<th>Stage of Process</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Good Deeds</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   c. How does the number of good deeds increase as the Pay It Forward process progresses in stages? How is that pattern of change shown in the plot of data?

   d. How many stages of this process will be needed before a total of 25,000 good deeds will be done?
3. In the two versions of Pay It Forward that you have studied, you can use the number of
good deeds at one stage to calculate the number at the next stage.
   a. Use the words NOW and NEXT to write recursive rules that express the two
      patterns.

   b. How do the numbers and calculations indicated in the rules express the patterns of
      change in tables of \((\text{stage, number of good deeds})\) data?

   c. Write a rule relating NOW and NEXT that could be used to model a Pay It Forward
      process in which each person does good deeds for four other new people. What
      pattern of change would you expect to see in a table of \((\text{stage, number of good deeds})\)
data for this Pay It Forward process?

4. What are the main steps (not keystrokes) required to use a calculator to produce tables
   of values like those you made in Problems 1 and 2?
Look back at the patterns of change in the number of good deeds in the different Pay it Forward schemes—three per person and two per person.

A. Compare the **PROCESSES** by noting similarities and differences in:

<table>
<thead>
<tr>
<th></th>
<th>i.) Patterns of change in the tables of data:</th>
<th>ii.) Patterns in the graphs of data:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SIMILARITIES</strong></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>DIFFERENCES</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SIMILARITIES</td>
<td>DIFFERENCES</td>
<td></td>
</tr>
<tr>
<td>------------------------------------------------------------------------------</td>
<td>------------------------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>iii.) Rules relating NOW and NEXT numbers of good deeds:</td>
<td>iv.) Rules expressing number of good deeds $N$ as a function of stage number $x$:</td>
<td></td>
</tr>
</tbody>
</table>
B. Compare **PATTERNS OF CHANGE** in numbers of good deeds at each stage of the Pay It Forward process to those of linear functions that you have studied in earlier work.

<table>
<thead>
<tr>
<th></th>
<th>i.) How are NOW-NEXT rules similar and how are they different?</th>
<th>ii.) How are ( y = ) rules similar and how are they different?</th>
<th>iii.) How are the patterns of change in tables and graphs of linear function similar and how are they different?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SIMILARITIES</strong></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>DIFFERENCES</strong></td>
<td></td>
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</tr>
</tbody>
</table>
The patterns in spread of good deeds by the Pay It Forward process occur in other quite different situations. For example, when bacteria infect some part of your body, they often grow and split into pairs of genetically equivalent cells over and over again.

a. Suppose a single bacterium lands in a cut on your hand. It begins spreading an infection by growing and splitting into two bacteria every 20 minutes.

i. Complete a table showing the number of bacteria after each 20-minute period in the first three hours. (Assume none of the bacteria are killed by white blood cells.)

<table>
<thead>
<tr>
<th>Number of 20-min Periods</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bacteria Count</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

ii. Plot the \((\text{number of time periods}, \text{bacteria count})\) values on your calculator and make a sketch below. Then describe the pattern of growth of bacteria causing the infection.

b. Use \(\text{NOW}\) and \(\text{NEXT}\) to write a rule relating the number of bacteria at one time to the number 20 minutes later. Then use the rule to find the number of bacteria after fifteen 20-minute periods.

c. Write a rule showing how the number of bacteria \(N\) can be calculated from the number of stages \(x\) in the growth and division process. Also write this rule in function notation.

d. How are the table, graph and symbolic rules describing bacteria growth similar to and different from the Pay It Forward examples? How are they similar to, and different from, typical patterns of linear functions?
The patterns of change that occur in counting the good deeds of a Pay It Forward scheme and the growing number of bacteria in a cut are examples of exponential growth. Observe the exponential functions rules we discovered in Investigation 1, why do you think such rules are considered exponential?

1. Many exponential start with a value other than 1. For example, infections seldom start with a single bacterium. In your groups, discuss different ways bacteria can enter into the human body.

   **Group 1:** Pick one approach and assume approximately 25 bacteria cells enter your body at that moment. Suppose also that the bacteria divide in two after every quarter of an hour.

   **Group 2:** Pick one approach and assume approximately 30 bacteria cells enter your body at that moment. Suppose also that the bacteria divide in two after every quarter of an hour.

   **Group 3:** Pick one approach and assume approximately 40 bacteria cells enter your body at that moment. Suppose also that the bacteria divide in two after every quarter of an hour.

   **Group 4:** Pick one approach and assume approximately 60 bacteria cells enter your body at that moment. Suppose also that the bacteria divide in two after every quarter of an hour.

   **Group 5:** Pick one approach and assume approximately 100 bacteria cells enter your body at that moment. Suppose also that the bacteria divide in two after every quarter of an hour.

   a. Make a table of \((\text{number of time periods, bacteria count})\) values for 8 quarter-hour time periods.

<table>
<thead>
<tr>
<th>Number of Quarter-Hour Periods</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Bacteria</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   b. Copy this table onto poster paper and plot the values on a coordinate grid.
c. What are reasonable questions an infected individual might have regarding the bacterial growth over time?

d. How could you go about answering such questions? How could you explain/show how you obtained your answers?
2. After observing the different scenarios from each group – what stayed the same in each problem and what changed?

How do we see these similarities and differenced in the tables, graphs, rules?

<table>
<thead>
<tr>
<th></th>
<th>Similarities</th>
<th>Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tables</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Graphs</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Function Notation Rules</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Recursive Rules</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Just as bacteria growth won’t always start with a single cell, other exponential growth processes can start with different initial numbers. Think again about the Pay It Forward scheme in Investigation 1.

3. Suppose that four good friends decide to start their own Pay It Forward tree. To start the tree, they each do good deeds for three different people. Each of those new people in the tree does good deeds for three other new people, and so on.

   a. What NOW-NEXT rule shows how to calculate the number of good deeds done at each stage of this tree?

   b. What “N = ...” rule shows how to calculate the number of good deeds done at any stage x of this tree? Also write this rule in function notation.

   c. How would the NOW-NEXT and “N = ...” rules be different if the group of friends starting the tree had five members instead of four?

   d. Which of the Pay It Forward schemes below would most quickly reach a stage in which 1,000 good deeds are done? Why does that scheme make sense?

   Scheme 1: Start with a group of four friends and have each person in the three do good deeds for two different people; or

   Scheme 2: Start with only two friends and have each person in the tree do good deeds for three other new people.

In studying exponential growth, it is helpful to know the initial value of the growing quantity. For example, the initial value of the growing bacteria population in Problem 1 was 25. You also need to know when the initial value occurs. For example, the bacteria population was 25 after 0 quarter-hour periods.

In Problem 4 on the other hand, 12 good deeds are done at Stage 1. In this context, “Stage 0” does not make much sense, but we can extend the pattern backward to reason that N = 4 when x = 0.
4. Use your calculator \( ^{\wedge} \) and the key to find each of the following values:
\( 2^0, 3^0, 5^0, 23^0 \).

   a. What seems to be the calculator value for \( b^0 \), for any positive value of \( b \)?

   b. Recall the examples of exponential patterns in bacterial growth. How do the “\( N = ... \)”
   rules for those situations make the calculator output for \( b^0 \) reasonable?

5. Suppose you are on team studying the growth of bacteria in a laboratory experiment. At
   the start of your work shift in the lab, there are 64 bacteria in one petri dish culture, and
   the population seems to be doubling every hour.

   a. What rule should predict the number of bacteria in the culture at a time \( x \) hours
      after the start of your work shift?

   b. What would it mean to calculate values of \( y \) for negative values of \( x \) in this situation?

   c. What value of \( y \) would you expect for \( x = -1 \)? For \( x = -2 \)? For \( x = -3 \) and \( -4 \)?

   d. Use your calculator to examine a table of \((x, y)\) values for the function \( y = 64(2^x) \)
      when \( x = 0, -1, -2, -3, -4, -5, -6 \). Compare your results to your experiment in
      Part c. Then explain how you could think about this problem of bacteria growth in a
      way so that the calculator results make sense.
6. Study the tables and graphs of \((x, y)\) values to estimate solutions for each of the following equations and inequalities. In each case, be prepared to explain what the solution tells about bacteria growth in the experiment of Problem 6.

a. \(1,024 = 64(2^x)\)  
b. \(8,192 = 64(2^x)\)  
c. \(64(2^x) > 25,000\)  
d. \(4 = 64(2^x)\)  
e. \(64(2^x) < 5,000\)  
f. \(64(2^x) = 32\)
The exponential functions that you studied in this investigation describe patterns of change in bacteria growth and numbers of people in a Pay it Forward tree. They have some features in common.

A and B.) What do the values of b and a tell about the pattern of change in represented by the \textbf{NOW-NEXT} and the \( y = a(b^x) \) rule? How will that pattern be illustrated in a table or graph of \((x, y)\) values?

<table>
<thead>
<tr>
<th>Pattern of Change</th>
<th>Next = (b \times \text{Now, starting at } a)</th>
<th>( y = a(b^x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tables</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Graphs</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
C. What is the value of $b^x$ when $x$ is 0? What could the result mean in a problem situation where exponential growth is being studied?

D. How would you calculate values of $b^x$ when $x$ is a negative number? What would those results mean in a problem situation where exponential growth is being studied?
The drug penicillin was discovered by observation of mold growing on biology lab dishes. Suppose a mold begins growing on a lab dish. When first observed, the mold covers 7 cm\(^2\) of the dish surface, but it appears to double in area every day.

a. What rules can be used to predict the area of the mold patch 4 days after the first measurement:
   i. using \(\text{NOW-NEXT} \) form?

   ii. using \( y=\ldots \) form?

b. How would each rule in Part a change if the initial mold area was only 3 cm\(^2\)?
   i. \(\text{NOW-NEXT} \) form:

   ii. \( y=\ldots \) form:

c. How would each rule in Part a change if the area of the mold patch increased by a factor of 1.5 every day?
   i. \(\text{NOW-NEXT} \) form:

   iii. \( y=\ldots \) form:

d. What mold area would be predicted after 5 days in each set of conditions from Parts a-c?

   (a.)

   (b.)

   (c.)
e. For “y=...” rules used in calculating growth of mold area, what would it mean to calculate values of $y$ when $x$ is a negative number?

f. Write and solve equations or inequalities that help to answer these questions.

i. If the area of the mold patch is first measured to be 5 cm$^2$ and the area doubles each day, how long will it take that mold sample to grow to an area of 40 cm$^2$?

ii. For how many days will the mold patch in part i have an area less than 330 cm$^2$?
Investigation 3

Every now and then you may hear about somebody winning a big payoff in a state lottery. The winnings can be 1, 5, 50 or even 400 million dollars. The big money wins are usually paid off in annual installments for about 20 years. But some smaller prizes are paid at once.

1. Imagine that you just won the daily lottery from a New York lottery ticket. You have two payoff choices.

   **Option 1:** Receive a single $10,000 payment now
   **Option 2:** A single payment of $20,000 ten year from now

   a. Discuss with others your thinking on which of the two payoff methods to choose.

   b. Suppose a local bank called and said you could invest your $10,000 payment in a special 10-year certificate of deposit (CD), earning 8% interest compounded yearly. How would this affect your choice of payoff method?

As you work on the problems in this investigation, look for answers to the question:

   *How can you represent and reason about functions involved in investments paying compound interest?*

Of the two lottery payoff methods, one has a value of $20,000 at the end of 10 years. The value (in 10 years) of receiving the $10,000 payoff now and putting it in a 1—year certificate of deposit paying 8% interest compounded annually is not so obvious.

- After one year, your balance will be:
  \[ 10,000 + (0.08 \times 10,000) = 1.08 \times 10,000 = 10,800 \]
- After the second year, your balance will be:
  \[ 10,800 + (0.08 \times 10,800) = 1.08 \times 10,800 = 11,664. \]

During the next year, the CD balance will increase in the same way, starting from $11,664, and so on.

2. Write rules that will allow you to calculate the balance of this certificate of deposit:
   a. for the next year, using the balance from the year before.

   b. after any number of years \( x \).
3. Use the rules in Problem 2 to determine the value of the certificate of deposit after 10 years. Then decide which 10-year plan will result in more money and how much more money that plan will provide.

4. Look for an explanation of your conclusion in Problem 3 by answering these questions about the potential value of the CD paying 8% interest compounded yearly.
   a. Describe the pattern of growth in the CD balance has time passes.
   
   b. Why isn’t the change in the CD balance the same each year?
   
   c. How is the pattern of increase in the CD balance shown in the shape of a graph for the function relating CD balance to time?
   
   d. How could the pattern of increase have been predicted by thinking about the rules (NOW-NEXT and “y = …”) relating CD balance to time?
5. Suppose that the prize winner decided to leave the money in the CD earning 8% interest for more than 10 years. Use tables or graphs to estimate solutions for the following equations or inequalities. In each case, be prepared to explain what the solution tells about the growth of a $10,000 investment that earns 8% interest compounded annually.

a. \(10,000(1.08^x) = 25,000\)  
   d. \(10,000(1.08^x) \geq 25,000\)

b. \(10,000(1.08^x) = 37,000\)  
   e. \(10,000(1.08^x) \leq 30,000\)

c. \(10,000(1.08^x) = 50,000\)  
   f. \(10,000(1.08^x) = 10,000\)
Most savings accounts operate in a manner similar to the bank’s certificate of deposit offer. However, they may have different starting balances, different interest rates, or different periods of investment.

A. Describe two ways to find the value of such a savings account at the end of each year from the start to year 10.

| i.) NOW-NEXT rule | ii.) \( y = a(b^x) \) rule |

B. What graph patterns would you expect from plots of (year, account balance) values?
C. How would the function rules change if the interest rate changes? If the initial investment changes?

D. Why does the dollar increase in the account balance get larger from each year to the next?

E. How are patterns of change that occur with the bank investment similar to and different from those of other functions that you've used while working on problems of Investigations 1 and 2? On problems of previous units?
From the Linear Functions Unit you recognized that most real world applications that model linear patterns are not perfectly linear. The same idea applies to real world applications involving exponential patterns. Can you think of other exponential patterns that we have yet to discuss?

As you work on the problems of this investigation, look for answers to the following questions:

*What are some useful strategies for finding functions modeling patterns of change that are only approximately exponential?*

1. Suppose that census counts of Midwest wolves begin in 1990 and produced these estimates for several different years:

<table>
<thead>
<tr>
<th>Time since 1990 (in years)</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated Wolf Population</td>
<td>100</td>
<td>300</td>
<td>500</td>
<td>900</td>
<td>1,500</td>
<td>3,100</td>
</tr>
</tbody>
</table>

   a. Plot the wolf population data in your calculator and decide whether a linear or exponential function seems to likely match the pattern of growth well. For the functions type of your choice, experiment with different rules to see which rule provides a good model of growth pattern.

   b. Use your calculator to find both linear and exponential regression models for the data pattern. Compare the fit of each function to that of the function you developed by experimentation in Part a.

   c. What do the numbers in the linear and exponential regression rules from Part b suggest about patterns of change in the wolf population?

   d. Use the model for wolf population growth that you believe to be best to calculate population estimates for the missing years 1994 and 2001 and then for the year 2015 and 2020.
2. Pick an animal of your choice to research and collect population data for and then fill out the table below;

<table>
<thead>
<tr>
<th>Time since _____ (in years)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated ________________ Population</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Plot the animal population data in your calculator and decide whether a linear or exponential function seems to likely match the pattern of growth well. For the functions type of your choice, experiment with different rules to see which rule provides a good model of growth pattern.

b. Use your calculator to find both linear and exponential regression models for the data pattern. Compare the fit of each function to that of the function you developed by experimentation in Part a.

c. What do the numbers in the linear and exponential regression rules from Part b suggest about patterns of change in the animal population?

d. Use the model for animal population growth that you believe to be best to calculate population estimates for four different missing years of your choice.

**Be prepared to share your results with the class**
A. How can you determine whether a data pattern is modeled best by a linear or an exponential function?

B. What do the numbers a and b in a linear function \( y = a + bx \) tell about patterns in:

<table>
<thead>
<tr>
<th>Graph</th>
<th>Table</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
C. What do the numbers \( c \) and \( d \) in an exponential function \( y = c(d^x) \) tell about patterns in:

<table>
<thead>
<tr>
<th>Graphs</th>
<th>Tables</th>
</tr>
</thead>
</table>

D. What strategies are available for finding a linear or exponential function that models a linear or exponential data pattern?
Write functions that provide good models for the patterns of change that relate $p$, $q$ and $r$ to $x$ in the following tables.

i.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-10</th>
<th>-5</th>
<th>0</th>
<th>6</th>
<th>15</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>12</td>
<td>15</td>
<td>18</td>
</tr>
</tbody>
</table>

ii.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-10</th>
<th>-5</th>
<th>0</th>
<th>6</th>
<th>15</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>1</td>
<td>8</td>
<td>60</td>
<td>650</td>
<td>25,000</td>
<td>190,000</td>
<td>11,000,000</td>
</tr>
</tbody>
</table>

iii.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-10</th>
<th>-5</th>
<th>0</th>
<th>6</th>
<th>15</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>1.0</td>
<td>1.3</td>
<td>1.6</td>
<td>2.25</td>
<td>3.4</td>
<td>4.4</td>
<td>7.0</td>
</tr>
</tbody>
</table>
Investigation 5

Recall that the **domain** of a function is the set of all numerical values that can be the input (x-values) of a function while the **range** of a function is the set of all numerical values that can be the output (y-values or f(x) values) of a function.

For example, consider the linear function \( f(x) = 2x + 1 \). What possible values can be put into this function so that an output can be found? In other words, what x-values will result in a corresponding y-value? What output values are possible?

Enter the function \( f(x) = 2x + 1 \) into your calculator. Graph the function in the standard window. A sketch of the graph is shown below in a smaller window. How does the graph help determine the possible input (domain) values? How does the graph help determine the possible output (range) values?

![Graph of f(x) = 2x + 1](image)

Create a table for the function \( f(x) = 2x + 1 \) on your calculator. How can the table help determine the possible input (domain) values? How does the table help determine the possible output (range) values?

When using the table to help determine the domain and range of a function, you must keep in mind that checking non-integer values of x may also be important.
Find the domain and range for each function below. First, think about what values make sense as inputs and as outputs. Then, enter the function rule into your calculator. Use the graph of the function and the corresponding table of values to help or to confirm your thinking.

1. \( f(x) = 3x - 2 \)
2. \( g(x) = 2^x \)

3. \( h(x) = 4 - 2x \)
4. \( j(x) = 5(3)^x \)

5. \( p(x) = |x - 3| \)
6. \( y = 5 \)
Chapter Four: Validity of Curriculum Project

The purpose of this chapter is to provide evidence and reflection of student of centered learning.

After collaboratively exploring mathematics for several weeks in small groups, twenty one Algebra students were given the following survey:

1. Do you feel as though the questions we explore in Algebra challenge you?

   Never  Hardly  Sometimes  Mostly  Always

2. Do you think the questions we explore advance your problem solving skills?

   Never  Hardly  Sometimes  Mostly  Always

3. Does small group work help in allowing you to understand different mathematical approaches?

   Never  Hardly  Sometimes  Mostly  Always

4. Do you find exploring math content in small groups supportive?

   Never  Hardly  Sometimes  Mostly  Always

5. Do you feel communicating mathematics helps you to better understand the content?

   Never  Hardly  Sometimes  Mostly  Always

6. If you have previously worked in a mathematics classroom where the teacher directly instructed you, please share your thoughts on which approach you prefer and why:
   1. small group/teacher as guide OR
   2. individual/teacher as leader
Students were given this survey during their thirteenth week of school. So far this year, students have spent Algebra working in small groups exploring math problems that have been designed to hopefully challenge the students enough to encourage collaboration and discussion. The survey was anonymous and students were encouraged to be completely honest within their answers.

1. **Do you feel as though the questions we explore in Algebra challenge you?**

   - Never
   - Hardly
   - Sometimes
   - Mostly
   - Always

<table>
<thead>
<tr>
<th></th>
<th>Never</th>
<th>Hardly</th>
<th>Sometimes</th>
<th>Mostly</th>
<th>Always</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>0</td>
<td>1</td>
<td>13</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

   This means that approximately 33% of these Algebra students felt as though the problems we explored in class were either always challenging or challenging for the majority of the time. 62% of the class felt an even balance of challenging yet obtainable problems. No one felt as though the questions we explore were overly easy. It is good to have balance of challenging yet manageable content.

2. **Do you think the questions we explore advance your problem solving skills?**

   - Never
   - Hardly
   - Sometimes
   - Mostly
   - Always

<table>
<thead>
<tr>
<th></th>
<th>Never</th>
<th>Hardly</th>
<th>Sometimes</th>
<th>Mostly</th>
<th>Always</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>12</td>
<td>5</td>
</tr>
</tbody>
</table>

42
From this data we can conclude that the majority of students, 81%, felt as though the questions we explored thus far in Algebra had enhanced their problem solving skills. The Algebra questions used are designed to encourage students to problem solve rather than regurgitate rules and steps taught to by the teacher.

3. **Does small group work help in allowing you to understand different mathematical approaches?**

<table>
<thead>
<tr>
<th></th>
<th>Never</th>
<th>Hardly</th>
<th>Sometimes</th>
<th>Mostly</th>
<th>Always</th>
</tr>
</thead>
<tbody>
<tr>
<td>Students</td>
<td>0</td>
<td>0</td>
<td>9</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

This data was rather shocking that only 57% of these students felt as though the group work helped them understand different mathematical approaches. This could be because they are only in the thirteenth week of school and still exploring the meaning of group work and haven’t gotten through the more challenging Algebra content. Once they reach this content, multiple approaches will be in essential technique to help them determine which approach they prefer.

4. **Do you find exploring math content in small groups supportive?**

<table>
<thead>
<tr>
<th></th>
<th>Never</th>
<th>Hardly</th>
<th>Sometimes</th>
<th>Mostly</th>
<th>Always</th>
</tr>
</thead>
<tbody>
<tr>
<td>Students</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>11</td>
<td>4</td>
</tr>
</tbody>
</table>
74% of students felt as though exploring math content in small groups was supportive. It would have been interesting to see first, what students have ever been taught the traditional way and what students have been working in small groups their entire math career.

5. Do you feel communicating mathematics helps you to better understand the content?

<table>
<thead>
<tr>
<th>Never</th>
<th>Hardly</th>
<th>Sometimes</th>
<th>Mostly</th>
<th>Always</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>9</td>
<td>9</td>
</tr>
</tbody>
</table>

Approximately 86% of students found that communicating mathematics better helps them to understand the content.

6. If you have previously worked in a mathematics classroom where the teacher directly instructed you, please share your thoughts on which approach you prefer and why:

1. small group/teacher as guide OR
2. individual/teacher as leader

Students were told that the above question was optional. They were encouraged to answer it if they have worked in a math class where they sat in rows, working alone, where the teacher stood in front and lectured. I gave them verbal directions to share which instruction they prefer and why. Students wrote the following:

- “I prefer to work in a small group because I can learn a lot more working with my peers. Instead of just having the teacher explain everything, my peers can help me.”
• “Small groups work with teacher as guider because I get ideas when in a small group and when people have questions it helps me understand better. I use more of my skill when in a small group.”

• “I prefer small group because it gets more brains involved and it does not cause as many problems.”

• “Small group – this is because as a group brainstorming is more broad and you can get though and understand a problem better opposed to a work place where there is one teacher and you individually work amongst yourself, so this means you become dependent on the teachers for answers.”

• “I prefer small groups because being lectured is BORING.”
Chapter Five: Conclusion

The paradigm shift from the NCTM standards to the Common Core State Standards provides opportunities for educators to transform their teaching approach. This curriculum project on Exponential Growth is designed to be a resource to help support this shift.

This curriculum is intended for students to be able to discover mathematics in a way they may have not done in the past. Students should be encouraged to explore real world contexts that allow them to hypothesize, problem solve and make conclusions in an ongoing process until they reach their final and correct summary. The teacher should act as a guide, pushing students prompting questions and guiding them in the right direction. The teacher is encouraged not to give on formulas for students to replicate and practice over and over again.

Students are encouraged to communicate the mathematics. This communication will allow students to understand and critique the work of others. Such communication will allow students to better understand the material and expand their thinking.

From student feedback, it is clear students believe communicating mathematics helps them to better understand the material. As educators, it is important we keep an open mind to change, and do what is best for student growth. Educators must allow students to engage collaboratively in open ended, real world contextual problems that allow students to hypothesize and draw conclusions on their own, rather than regurgitating rules and definitions modeled by an instructor.
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http://www.corestandards.org/standards-in-your-state/


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Appendix

Exponential Growth Answer Key

Think About It…

Continue Trevor’s kind of Pay It Forward thinking.

a) How many people would receive a Pay It Forward good deed at each of the next several stages of the process?

The first 6 stages would yield the following numbers of good deeds: 3, 9, 27, 81, 243, and 729.

b) What is your best guess about the number of people who would receive Pay It Forward good deeds at the tenth stage of the process?

Students should only be guessing the number of good deeds. They are not expected to have well-practiced strategies for finding the answer. (The 10th stage would yield \(3^{10} = 59,059\) good deeds.)

c) Which of the graphs do you think is most likely to represent the pattern by which the number of people receiving Pay It Forward good deeds increases as the process continues over time? Describe the pattern of change for each graph.

Graph III represents the good deed pattern best because it shows increase at an increasing rate. Students may suggest the other graphs at this time and that is okay. The important part is that they should be able to justify their answer and how their choice matches the context.
Investigation 1

The number of good deeds in the Pay It Forward pattern can be represented by a tree graph that starts like this:

1. At the start of the Pay It Forward process, only one person does good deeds – for three new people. In the next stage, the three new people each do good things for three more new people. In the next stage, nine people each do good things for three more new people, and so on, with no person receiving more than one good deed.

   a. Make a table that shows the number of people who will receive good deeds at each of the next seven stages of the Pay It Forward process. Then plot the (stage, number of good deeds) data in your calculator.

<table>
<thead>
<tr>
<th>Stage of Process</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Good Deeds</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td>81</td>
<td>243</td>
<td>729</td>
<td>2187</td>
<td>6561</td>
<td>19683</td>
<td>59049</td>
</tr>
</tbody>
</table>

   b. How does the number of good deeds at each stage grow as the tree progresses? How is that pattern of change shown in the plot of the data?

   The number of good deeds triples from one stage to the next. This pattern of change is shown in the plot by the increase in the vertical distance from x-axis and also between data points. Reading along the x-axis, the vertical distance from the x-axis and from one point to the next point, increases by a multiple of 3. This understanding is not expected from the students at this point – they should be able to recognize that the increasing rate is shown in the plot by the graph being level at first but then becomes steeper and steeper as you move further along the x-axis.

   c. How many stages of the Pay It Forward process will be needed before a total of at least 25,000 good deeds will be done?

   Students may need help recognizing that the table entries do not give cumulative totals. It might be helpful to ask some prompting questions to get students to recognize this, then have them make a 3rd column on their table.

   After the ninth stage, a total of 29,523 good deeds will have been done.
2. Consider now how the number of good deeds would grow if each person touched by the Pay It Forward process were to do good deeds for only two other new people, instead of three.

   a. Make a tree graph for several stages of this Pay It Forward process.

   ![Tree Diagram](image)

   b. Make a table showing the number of good deeds increase as the Pay It Forward process progress and plot those sample (stage, number of good deeds) values.

<table>
<thead>
<tr>
<th>Stage of Process</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Good Deeds</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td>128</td>
<td>256</td>
<td>512</td>
<td>1024</td>
</tr>
</tbody>
</table>

   c. How does the number of good deeds increase as the Pay It Forward process progresses in stages? How is that pattern of change shown in the plot of data?

   *Each new stage yields twice the number of good deeds as the previous stage. This makes for a pattern of growth that increases at an increasing rate and a plot starts off fairly level, and then becomes steeper and steeper as time increases.*

   d. How many stages of this process will be needed before a total of 25,000 good deeds will be done?

   *At the 14th stage, a total of 32,766 good deeds will have been done.*
3. In the two versions of Pay It Forward that you have studied, you can use the number of good deeds at one stage to calculate the number at the next stage.
   a. Use the words NOW and NEXT to write recursive rules that express the two patterns.

\[
\text{Problem 1: } \text{Next} = \text{Now} \cdot 3 \quad \text{starting at 1}
\]
\[
\text{Problem 2: } \text{Next} = \text{Now} \cdot 2 \quad \text{starting at 1}
\]

b. How do the numbers and calculations indicated in the rules express the patterns of change in tables of (stage, number of good deeds) data?

*In the tables, each number of good deeds is either 3 times or 2 times the number at the previous stage.*

c. Write a rule relating NOW and NEXT that could be used to model a Pay It Forward process in which each person does good deeds for four other new people. What pattern of change would you expect to see in a table of (stage, number of good deeds) data for this Pay It Forward process?

\[
\text{Next} = \text{Now} \cdot 4 \quad \text{starting at 4}
\]

*The pattern of change would be increasing at an increasing rate.*

4. What are the main steps (not keystrokes) required to use a calculator to produce tables of values like those you made in Problems 1 and 2?

*Enter the first number of good deeds in your in the calculator. Then use that answer times 3 for each stage.*
Look back at the patterns of change in the number of good deeds in the different Pay it Forward schemes—three per person and two per person.

A. Compare the PROCESSES by noting similarities and differences in:

<table>
<thead>
<tr>
<th>SIMILARITIES</th>
<th>DIFFERENCES</th>
</tr>
</thead>
<tbody>
<tr>
<td>i.) Patterns of change in the tables of data:</td>
<td>ii.) Patterns in the graphs of data:</td>
</tr>
<tr>
<td><em>In both cases multiply the previous value by the number of good deeds to get to the next value</em></td>
<td><em>As x increases by 1, y increases at an increasing rate</em></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>When x was 1, for 2 good deeds, y=2</td>
<td>For 3 good deeds, the graph got steeper faster – this is because you multiply by a higher number cause larger products</td>
</tr>
<tr>
<td>When x was 1 for 3 good deeds, y=3</td>
<td>For 2 good deeds, the graph got steeper slower</td>
</tr>
<tr>
<td>x 1 2 3 4</td>
<td>x 1 2 3 4</td>
</tr>
<tr>
<td>y 2 4 8 16</td>
<td>y 3 9 27 81</td>
</tr>
</tbody>
</table>
### iii.) Rules relating NOW and NEXT numbers of good deeds:

- *In both you multiply NOW by the number of good deeds performed at each stage*
- *In both, the starting at value was 1*

### iv.) Rules expressing number of good deeds $N$ as a function of stage number $x$:

- *Both have $x$ as the exponent*
  - $y = 3^x$ or $y = 2^x$
  - $f(x) = 3^x$ or $f(x) = 2^x$
- *Base number represents # of good deeds*

---

<table>
<thead>
<tr>
<th><strong>SIMILARITIES</strong></th>
<th><strong>DIFFERENCES</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>DIFFERENCES</strong></td>
</tr>
</tbody>
</table>
|                   | *Different number of good deeds*  
  *Next = Now \cdot \_?\_?*  
  # of good deeds | *Different base  
  or  
  # of good deeds* |
B. Compare **PATTERNS OF CHANGE** in numbers of good deeds at each stage of the Pay It Forward process to those of linear functions that you have studied in earlier work.

<table>
<thead>
<tr>
<th>SIMILARITIES</th>
<th>i.) How are NOW-NEXT rules similar and how are they different?</th>
<th>ii.) How are y= rules similar and how are they different?</th>
<th>iii.) How are the patterns of change in tables and graphs of linear function similar and how are they different?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Both start with Next = Now</td>
<td>Both start with y= Can see the rate of change in both values</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Both have a starting at value</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Both are used to determine future values</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>DIFFERENCES</th>
<th>Exponential: NOW-NEXT rules require Multiplication</th>
<th>y = mx+b</th>
<th>Exponential – As x increases by 1, y increases at an increasing rate *Multiply by the constant to get to the next value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear: NOW-NEXT rules require Addition/Subtraction</td>
<td>y = bx</td>
<td>Exponential rules have an exponent of x</td>
<td>Linear – As x increases by 1, y increases at a constant rate *Add the constant to get from one value to the next</td>
</tr>
</tbody>
</table>

| | | | |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| | | | | |
a. Suppose a single bacterium lands in a cut on your hand. It begins spreading an infection by growing and splitting into two bacteria every 20 minutes.

i. Complete a table showing the number of bacteria after each 20-minute period in the first three hours. (Assume none of the bacteria are killed by white blood cells.

<table>
<thead>
<tr>
<th>Number of 20-min Periods</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bacteria Count</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td>128</td>
<td>256</td>
<td>512</td>
</tr>
</tbody>
</table>

ii. Plot the (number of time periods, bacteria count) values on your calculator then describe the pattern of growth of bacteria causing the infection.

b. Use NOW and NEXT to write a rule relating the number of bacteria at one time to the number 20 minutes later. Then use the rule to find the number of bacteria after fifteen 20-minute periods.  
   Next = Now (2) starting at 2
   After fifteen 20-minute periods, there would be 32,768 bacteria

c. Write a rule showing how the number of bacteria $N$ can be calculated from the number of stages $x$ in the growth and division process. Also write this rule in function notation.
   $N = 2^x$ or $N(x) = 2^x$

d. How are the table, graph and symbolic rules describing bacteria growth similar to and different from the Pay It Forward examples? How are they similar to, and different from, typical patterns of liner functions?
   Tables, graphs are rules are all identical – however the $y$ values represent different variables.
   In the tables for linear functions, you add/subtract to get to the next $y$-values, for exponential; you multiply to get to the next $y$-values.
   Linear graphs are lines whereas exponential graphs curve
The patterns of change that occur in counting the good deeds of a Pay It Forward scheme and the growing number of bacteria in a cut are examples of exponential growth. Observe the exponential functions rules we discovered in Investigation 1, why do you think such rules are considered exponential?

Because they always have the independent variable \((x)\) as the exponent

1. Many exponential start with a value other than 1. For example, infections seldom start with a single bacterium. In your groups, discuss different ways bacteria can enter into the human body.

**Group 1:** Pick one approach and assume approximately 25 bacteria cells enter your body at that moment. Suppose also that the bacteria divide in two after every quarter of an hour.

**Group 2:** Pick one approach and assume approximately 30 bacteria cells enter your body at that moment. Suppose also that the bacteria divide in two after every quarter of an hour.

**Group 3:** Pick one approach and assume approximately 40 bacteria cells enter your body at that moment. Suppose also that the bacteria divide in two after every quarter of an hour.

**Group 4:** Pick one approach and assume approximately 60 bacteria cells enter your body at that moment. Suppose also that the bacteria divide in two after every quarter of an hour.

**Group 5:** Pick one approach and assume approximately 100 bacteria cells enter your body at that moment. Suppose also that the bacteria divide in two after every quarter of an hour.

a. Make a table of \((\text{number of time periods}, \text{bacteria count})\) values for 8 quarter-hour time periods.

<table>
<thead>
<tr>
<th>Group 1</th>
<th>Number of Quarter-Hour Periods</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of Bacteria</td>
<td>25</td>
<td>50</td>
<td>100</td>
<td>200</td>
<td>400</td>
<td>800</td>
<td>1600</td>
<td>3200</td>
<td>6400</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Group 2</th>
<th>Number of Quarter-Hour Periods</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of Bacteria</td>
<td>30</td>
<td>60</td>
<td>120</td>
<td>240</td>
<td>480</td>
<td>960</td>
<td>1920</td>
<td>3840</td>
<td>7680</td>
</tr>
</tbody>
</table>
b. Copy this table onto poster paper and plot the values on a coordinate grid.

<table>
<thead>
<tr>
<th>GROUP 3</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>0 1 2 3 4 5 6 7 8</td>
</tr>
<tr>
<td>of hour-</td>
<td>Hour Periods</td>
</tr>
<tr>
<td>number</td>
<td></td>
</tr>
<tr>
<td>bacteria</td>
<td>40 80 160 320 640 1280 2560 5120 10240</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>GROUP 4</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>0 1 2 3 4 5 6 7 8</td>
</tr>
<tr>
<td>of hour-</td>
<td>Hour Periods</td>
</tr>
<tr>
<td>number</td>
<td></td>
</tr>
<tr>
<td>bacteria</td>
<td>60 120 240 480 960 1920 3840 7680 15360</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>GROUP 5</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>0 1 2 3 4 5 6 7 8</td>
</tr>
<tr>
<td>of hour-</td>
<td>Hour Periods</td>
</tr>
<tr>
<td>number</td>
<td></td>
</tr>
<tr>
<td>bacteria</td>
<td>100 200 400 800 1600 3200 6400 12800 25600</td>
</tr>
</tbody>
</table>

c. What are reasonable questions an infected individual might have regarding the bacterial growth over time?

   How long will it take to reach ______ bacteria?
   At ______ quarter hour periods, how many bacteria will be in the system?

d. How could you go about answering such questions? How could you explain/show how you obtained your answers?

   1. Continue on the table (use the table in your calc.)
   2. Plot the values in a graph in your calculator
   3. Create a y= rule (be sure students come up with this rule and share in group presentations)
   4. Create a recursive rule (be sure students come up with this recursive rule and share in group presentations)
2. After observing the different scenarios from each group – what stayed the same in each problem and what changed?

The starting at number of bacteria changed from group to group but the bacteria remained to double from group to group

How do we see these similarities and differences in the tables, graphs, rules?

<table>
<thead>
<tr>
<th></th>
<th>Similarities</th>
<th>Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tables</td>
<td><em>In order to get from the current y-value to the next, you multiply by 2</em></td>
<td><em>When x=0, y was the initial amounts of bacteria</em></td>
</tr>
<tr>
<td>Graphs</td>
<td><em>All graphs maintained the same steepness</em></td>
<td><em>Graphs had different y-intercepts (all were the initial amounts of bacteria)</em></td>
</tr>
<tr>
<td>Function</td>
<td><em>All in the form y = a(b)^x</em></td>
<td><em>The a value changed depending on the initial bacteria amount</em></td>
</tr>
<tr>
<td>Notation Rules</td>
<td><em>The b value was always 2</em></td>
<td></td>
</tr>
<tr>
<td>Recursive</td>
<td><em>Next = Now × 2</em></td>
<td><em>The starting at values were different</em></td>
</tr>
<tr>
<td>Rules</td>
<td><em>Starting at: initial amount of bacteria</em></td>
<td></td>
</tr>
</tbody>
</table>
Just as bacteria growth won’t always start with a single cell, other exponential growth processes can start with different initial numbers. Think again about the Pay It Forward scheme in Investigation 1.

3. Suppose that four good friends decide to start their own Pay It Forward tree. To start the tree, they each do good deeds for three different people. Each of those new people in the tree does good deeds for three other new people, and so on.

   a. What $\text{NOW-NEXT}$ rule shows how to calculate the number of good deeds done at each stage of this tree?

   \[ \text{Next} = \text{Now} \times 3, \text{starting at 12 good deeds} \]

   b. What “$N = \ldots$” rule shows how to calculate the number of good deeds done at any stage $x$ of this tree? Also write this rule in function notation.

   \[ N = 4(3^x) \text{ or } N(x) = 4(3^x) \]

   c. How would the $\text{NOW-NEXT}$ and “$N = \ldots$” rules be different if the group of friends starting the tree had five members instead of four?

   The starting value for the $\text{NOW-NEXT}$ rule would be $3(5) = 15$, rather than $3(4) = 12$. In the “$N=\ldots$” rules, the coefficient multiples by $3^x$ would be 5 rather than 4.

   d. Which of the Pay It Forward schemes below would most quickly reach a stage in which 1,000 good deeds are done? Why does that scheme make sense?

   **Scheme 2:** Start with only two friends and have each person in the tree do good deeds for three other new people.

   **Scheme 1:** 1,024 good deeds at stage 6 and **Scheme 2:** 1,458 good deeds at stage 6

   The results for scheme 2 are greater than those for scheme 1 in every stage beyond the first. This is reasonable because the size of the exponential base is much more influential in shaping the pattern of the growth than the starting value.

   In studying exponential growth, it is helpful to know the initial value of the growing quantity. For example, the initial value of the growing bacteria population in Problem 1 was 25. You also need to know when the initial value occurs. For example, the bacteria population was 25 after 0 quarter-hour periods.

   In Problem 4 on the other hand, 12 good deeds are done at Stage 1. In this context, “Stage 0” does not make much sense, but we can extend the pattern backward to reason that $N = 4$ when $x = 0$. 

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4. Use your calculator and the key to find each of the following values: \(2^0, 3^0, 5^0, 23^0\).

   a. What seems to be the calculator value for \(b^0\), for any positive value of \(b\)?

   \[\text{The result always is 1}\]

   b. Recall the examples of exponential patterns in bacterial growth. How do the “\(N = \ldots\)” rules for those situations make the calculator output for \(b^0\) reasonable?

   \[\text{At stage 0, or x values of 0, the bacteria rules give the initial amounts of bacteria since } b^0 = 1 \text{ and is multiplied by the initial amount of bacteria, a.}\]

5. Suppose you are on team studying the growth of bacteria in a laboratory experiment. At the start of your work shift in the lab, there are 64 bacteria in one petri dish culture, and the population seems to be doubling every hour.

   a. What rule should predict the number of bacteria in the culture at a time \(x\) hours after the start of your work shift?

   \[N(x) = 64(2^x)\]

   b. What would it mean to calculate values of \(y\) for negative values of \(x\) in this situation?

   \[\text{Use negative values of x would mean the number of hours previous to the starting point of the experiment. The y values would then represent the number of bacteria at previous time periods.}\]

   c. What value of \(y\) would you expect for \(x = -1\)? For \(x = -2\)? For \(x = -3\) and \(-4\)?

   \[\text{For } x = -1, -2, -3, -4, \ldots, \text{ corresponding y values are } 32, 16, 8, 4, \ldots \ .\]

   d. Use your calculator to examine a table of \((x, y)\) values for the function \(y = 64(2^x)\) when \(x = 0, -1, -2, -3, -4, -5, -6\). Compare your results to your experiment in Part c. Then explain how you could think about this problem of bacteria growth in a way so that the calculator results make sense.

   \[
   \begin{array}{ccccccc}
   x & -6 & -5 & -4 & -3 & -2 & -1 \\
   y & 1 & 2 & 4 & 8 & 16 & 32 \\
   \end{array}
   \]

   \[\text{Since moving forward in time produces doubling of the number of bacteria, moving backward should reasonable involve multiple the current amounts by one-half.}\]
6. Study the tables and graphs of \((x,y)\) values to estimate solutions for each of the following equations and inequalities. In each case, be prepared to explain what the solution tells about bacteria growth in the experiment of Problem 6.

a. \(1,024 = 64(2^x)\)  
b. \(8,192 = 64(2^x)\)  
c. \(64(2^x) > 25,000\)  
d. \(4 = 64(2^x)\)  
e. \(64(2^x) < 5,000\)  
f. \(64(2^x) = 32\)

\[
a. \ x = 4; \text{ the bacteria population is 1,024 after 4 hours.} \\
b. \ x = 7; \text{ the bacteria population is 8,192 after 7 hours.} \\
c. \ x \geq 9; \text{ the bacteria population will be more than 25,000 after 9 or more hours.} \\
   \text{(Assuming nearly continuous growth, a more precise estimate would be } x > 8.61 \text{ hours)} \\
d. \ x = -4; \text{ four hours before the experiment, the bacteria population was 4} \\
e. \ x \leq 6; \text{ the bacteria population will be less than 5,000 for every time amount less than or equal to 6 hours. (Assuming nearly continuous growth, a more precise estimate would be 6.288 hours.)} \\
f. \ x = -1; \text{ the population was 32,1 hours before the first count was made.}
\]
The exponential functions that you studied in this investigation describe patterns of change in bacteria growth and numbers of people in a Pay it Forward tree. They have some features in common.

A and B.) What do the values of \( b \) and \( a \) tell about the pattern of change in represented by the **NOW-NEXT** and the \( y = a(b^x) \) rule? How will that pattern be illustrated in a table or graph of \((x, y)\) values?

<table>
<thead>
<tr>
<th>Pattern of Change</th>
<th>Next = ( b \cdot \text{Now}, \text{starting at} \ a )</th>
<th>( y = a(b^x) )</th>
</tr>
</thead>
</table>
| **Tables** | • ‘\( a \)’ when \( x = 0, y = a \) | \( a \): \[
\begin{array}{c|c}
 x & 0 \\
\hline
 y & a \\
\end{array}
\]
• ‘\( b \)’ is the number you are multiplying by to get to the next \( y \)-value | \[
\begin{array}{c|c|c|c|c}
 x & 0 & 1 & 2 \\
\hline
 y & a & ab & a\cdot b \\
\end{array}
\]
\( x=0 \quad y=a \quad ab \quad a\cdot b \)
| **Graphs** | • \( a \) is the \( y \)-intercept | • \( b \) tells you how the \( y \) value will change (ex) triple, double, etc.
○ The larger the value of \( b \), the steeper the graph |
C. What is the value of \( b^x \) when \( x \) is 0? What could the result mean in a problem situation where exponential growth is being studied?

\[
b^0 = 1
\]

This means 1 is the starting of \( \text{__________ (deeds/bacteria/etc)} \)

\[y = a(b^x) \rightarrow a(b^0) \rightarrow a(1) \rightarrow a\]

Starts at \( a \)

D. How would you calculate values of \( b^x \) when \( x \) is a negative number? What would those results mean in a problem situation where exponential growth is being studied?

Work backwards by dividing by that number
The drug penicillin was discovered by observation of mold growing on biology lab dishes. Suppose a mold begins growing on a lab dish. When first observed, the mold covers 7 cm² of the dish surface, but it appears to double in area every day.

a. What rules can be used to predict the area of the mold patch 4 days after the first measurement:

   iii. using NOW-NEXT form? \[ \text{NEXT} = \text{NOW} \times 2 \]
       \[ \text{STARTING AT 7} \]

   iv. using “y=...” form? \[ y = 7(2)^x \]

b. How would each rule in Part a change if the initial mold area was only 3 cm²?

   iv. NOW-NEXT form: \[ \text{NEXT} = \text{NOW} \times 2 \]
      \[ \text{STARTING AT 3} \]

   v. “y=...” form: \[ y = 3(2)^x \]

c. How would each rule in Part a change if the area of the mold patch increased by a factor of 1.5 every day?

   i. NOW-NEXT form: \[ \text{NEXT} = \text{NOW} \times 1.5 \]
      \[ \text{STARTING AT 7} \]

   vi. “y=...” form: \[ y = 7(1.5)^x \]

d. What mold area would be predicted after 5 days in each set of conditions from Parts a-c?

   (a.) \[ 224 \text{ cm}^2 \]

   (b.) \[ 96 \text{ cm}^2 \]

   (c.) \[ 53.15625 \text{ cm}^2 \]
e. For “y=...” rules used in calculating growth of mold area, what would it mean to calculate values of y when x is a negative number?

*The amount of mold for times before the first observation*

f. Write and solve equations or inequalities that help to answer these questions.

iii. If the area of the mold patch is first measured to be 5 cm\(^2\) and the area doubles each day, how long will it take that mold sample to grow to an area of 40 cm\(^2\)?

\[
40 = 5(2)^x
\]

\[
x = 3
\]

3 days

iv. For how many days will the mold patch in part i have an area less than 330 cm\(^2\)?

\[
5(2)^x < 330
\]

\[
x \leq 6
\]

For 6 days
Every now and then you may hear about somebody winning a big payoff in a state lottery. The winnings can be 1, 5, 50 or even 400 million dollars. The big money wins are usually paid off in annual installments for about 20 years. But some smaller prizes are paid at once.

1. Imagine that you just won the daily lottery from a New York lottery ticket. You have two payoff choices.

   **Option 1:** Receive a single $10,000 payment now
   **Option 2:** A single payment of $20,000 ten year from now

   a. Discuss with others your thinking on which of the two payoff methods to choose.

   b. Suppose a local bank called and said you could invest your $10,000 payment in a special 10-year certificate of deposit (CD), earning 8% interest compounded yearly. How would this affect your choice of payoff method?

   *Compared to what banks offer, this would be a great interest rate*

As you work on the problems in this investigation, look for answers to the question:

*How can you represent and reason about functions involved in investments paying compound interest?*

Of the two lottery payoff methods, one has a value of $20,000 at the end of 10 years. The value (in 10 years) of receiving the $10,000 payoff now and putting it in a 1-year certificate of deposit paying 8% interest compounded annually is not so obvious.

- After one year, your balance will be:
  \[ 10,000 + (0.08 \times 10,000) = 1.08 \times 10,000 = 10,800 \]
- After the second year, your balance will be:
  \[ 10,800 + (0.08 \times 10,800) = 1.08 \times 10,800 = 11,664. \]

During the next year, the CD balance will increase in the same way, starting from $11,664, and so on.

2. Write rules that will allow you to calculate the balance of this certificate of deposit:
   a. for the next year, using the balance from the year before.

   \[ \text{NEXT} = \text{NOW} \times (1.08), \text{ starting at 10,000} \]

   b. after any number of years \( x \).

   \[ y = 10000(1.08)^x \]
3. Use the rules in Problem 2 to determine the value of the certificate of deposit after 10 years. Then decide which 10-year plan will result in more money and how much more money that plan will provide.

   After 10 years the deposit will grow to $21,589.25

   Therefore it is $10,000 more and the better deal

4. Look for an explanation of your conclusion in Problem 3 by answering these questions about the potential value of the CD paying 8% interest compounded yearly.
   a. Describe the pattern of growth in the CD balance has time passes.

      As the number of years increases by one, the CD balance increases at an increasing rate

   b. Why isn’t the change in the CD balance the same each year?

      The change is different from one year to the next because even though you are continuously multiplying by 1.08, the number you are multiplying the 1.08 by keeps changing. The bigger that number is, the bigger the change.

   c. How is the pattern of increase in the CD balance shown in the shape of a graph for the function relating CD balance to time?

      The graph appears to be flat in the beginning and then the curve gets steeper and steeper as time increases.

   d. How could the pattern of increase have been predicted by thinking about the rules (NOW-NEXT and “y = …”) relating CD balance to time?

      Models the same general format.

      However, this growth rate is much smaller so the graph has less curvature
5. Suppose that the prize winner decided to leave the money in the CD earning 8% interest for more than 10 years. Use tables or graphs to estimate solutions for the following equations or inequalities. In each case, be prepared to explain what the solution tells about the growth of a $10,000 investment that earns 8% interest compounded annually.

a. \(10,000(1.08^x) = 25,000\)
   
   **Nearest whole number** \(x = 12\) or \(x = 11.906\) getting more precise using different table/graph settings in calculator
   
   After 12 years the certificate is worth about $25,000

b. \(10,000(1.08^x) = 37,000\)

   **Nearest whole number** \(x = 17\)
   
   After 17 years, the certificate is worth about $37,000

c. \(10,000(1.08^x) = 50,000\)

   **Nearest whole number** \(x = 21\) or \(x = 20.912\)
   
   After 21 years the certificate is worth about $50,000

d. \(x \geq 12\)

   The CD is worth more than $25,000 at 12 years and beyond

e. \(x \leq 14\) or \(x < 15\) – more precisely \(x \leq 14.274\)

   The CD is worth less than $30,000 for up to 15 years

f. \(x = 0\)

   The CD is worth $10,000 at the time of the investment
Most savings accounts operate in a manner similar to the bank’s certificate of deposit offer. However, they may have different starting balances, different interest rates, or different periods of investment.

A. Describe two ways to find the value of such a savings account at the end of each year from the start to year 10.

i.) **NOW-NEXT rule**

Multiply the starting balance by the sum of 1 and the interest rate. Continue to multiply each new answer by the sum of 1 and the interest rate

\[
NEXT = NOW \times (1 + \text{interest rate})
\]

Starting at the initial deposit amount

ii.) **\(y = a(b^x)\) rule**

Substitute the initial deposit amount in for the ‘a’ value

Substitute the sum of 1 and the interest rate in for the ‘b’ value

B. What graph patterns would you expect from plots of (year, account balance) values?

As time increases, the account balance will always increase at an increasing rate

The y-intercept will always be the initial account balance
C. How would the function rules change if the interest rate changes? If the initial investment changes?

\[ y = a(b)^x \]

*If the interest rate changes – the b value will change*

*If the initial investment changes – the a value will change*

D. Why does the dollar increase in the account balance get larger from each year to the next?

*The dollar increase gets larger from year to year because each additional year the account balance that earns interest is itself increasing. The amount of increase is a set percentage being taken of larger and larger amounts, thus leading to an increasing rate of change*

E. How are patterns of change that occur with the bank investment similar to and different from those of other functions that you’ve used while working on problems of Investigations 1 and 2? On problems of previous units?

*Similarities:*
  - The good deed process and bacteria situations – all patterns were nonconstant and all were increasing at an increasing rate.

*Differences:*
  - The bank patterns was not as much of a drastic increase due to the small percentage of increase*
1. Suppose that census counts of Midwest wolves begin in 1990 and produced these estimates for several different years:

<table>
<thead>
<tr>
<th>Time since 1990 (in years)</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated Wolf Population</td>
<td>100</td>
<td>300</td>
<td>500</td>
<td>900</td>
<td>1,500</td>
<td>3,100</td>
</tr>
</tbody>
</table>

   a. Plot the wolf population data in your calculator and decide whether a linear or exponential function seems to likely match the pattern of growth well. For the functions type of your choice, experiment with different rules to see which rule provides a good model of growth pattern.

   b. Use your calculator to find both linear and exponential regression models for the data pattern. Compare the fit of each function to that of the function you developed by experimentation in Part a.

   \[ y = 209x - 284 \]
   \[ y = 186(1.19)^x \rightarrow \text{Exponential is better fit} \]

   c. What do the numbers in the linear and exponential regression rules from Part b suggest about patterns of change in the wolf population?

   \[ \text{Linear} - \text{The wolf population in 1990 is } -284 \text{ wolves and every year the population increases by 209 wolves.} \]
   \[ \text{Exponential} - \text{The wolf population in 1990 is 186 and every year the population grows by 19\%} \]

   d. Use the model for wolf population growth that you believe to be best to calculate population estimates for the missing years 1994 and 2001 and then for the year 2015 and 2020.

   \[ \text{Using the exponential rule:} \]
   \[ 1994: \text{approximately 372 wolves} \]
   \[ 2001: \text{approximately 1260 wolves} \]
   \[ 2015: \text{approximately 14394 wolves} \]
   \[ 2020: \text{approximately 34349 wolves} \]
2. Pick an animal of your choice to research and collect population data for and then fill out the table below;

<table>
<thead>
<tr>
<th>Time since _____ (in years)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated ____________ Population</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Plot the animal population data in your calculator and decide whether a linear or exponential function seems to likely match the pattern of growth well. For the functions type of your choice, experiment with different rules to see which rule provides a good model of growth pattern.

b. Use your calculator to find both linear and exponential regression models for the data pattern. Compare the fit of each function to that of the function you developed by experimentation in Part a.

c. What do the numbers in the linear and exponential regression rules from Part b suggest about patterns of change in the animal population?

d. Use the model for animal population growth that you believe to be best to calculate population estimates for four different missing years of your choice.

*Answers in the problem will vary. It is important you help students in choosing good resources. Allow students time to share their findings to the class once finished.*
A. How can you determine whether a data pattern is modeled best by a linear or an exponential function?

Consider the context of the problem
Look at a table of values – if the rate of change is approximately constant or not
Plot points – draw in a model that best fits the data

B. What do the numbers a and b in a linear function $y=a + bx$ tell about patterns in:

<table>
<thead>
<tr>
<th>Graph</th>
<th>Table</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$ – tells you the where the graph crosses the y-axis</td>
<td>$a$ – the value of y when $x = 0$</td>
</tr>
<tr>
<td>$b$ – tells you the slope of the graph (the larger the absolute value of $b$ is, the steeper the graph)</td>
<td>$b$ – what the y values increase/decrease by as $x$ increases by one</td>
</tr>
</tbody>
</table>
C. What do the numbers $b$ and $b$ in an exponential function $y = a(b^x)$ tell about patterns in:

<table>
<thead>
<tr>
<th>Graphs</th>
<th>Tables</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$ – tells you the where the graph crosses the $y$-axis</td>
<td>$a$ – the value of $y$ when $x = 0$</td>
</tr>
<tr>
<td>$b$ – how steep/quickly the graph is increasing. The larger the value of $b$, the steeper the graph will become</td>
<td>$b$ – the number you multiply the previous $y$ value by to get to the next $y$ value (as $x$ is increasing by one)</td>
</tr>
</tbody>
</table>

E. What strategies are available for finding a linear or exponential function that models a linear or exponential data pattern?

*Linear Regression or Exponential Regression on Calculator*
CHECK YOUR UNDERSTANDING

Write functions that provide good models for the patterns of change that relate $p$, $q$ and $r$ to $x$ in the following tables.

i. $\begin{array}{|c|c|c|c|c|c|c|c|} \hline x & -10 & -5 & 0 & 6 & 15 & 20 & 30 \\
p & 1 & 3 & 5 & 8 & 12 & 15 & 18 \\ \hline \end{array}$

$p = 0.44x + 5.32$

ii. $\begin{array}{|c|c|c|c|c|c|c|c|} \hline x & -10 & -5 & 0 & 6 & 15 & 20 & 30 \\
q & 1 & 8 & 60 & 650 & 25,000 & 190,000 & 11,000,000 \\ \hline \end{array}$

$q = 58.615(1.498)^x$

iii. $\begin{array}{|c|c|c|c|c|c|c|c|} \hline x & -10 & -5 & 0 & 6 & 15 & 20 & 30 \\
r & 1.0 & 1.3 & 1.6 & 2.25 & 3.4 & 4.4 & 7.0 \\ \hline \end{array}$

$r = 1.64(1.05)^x$
Recall that the **domain** of a function is the set of all numerical values that can be the input (x-values) of a function while the **range** of a function is the set of all numerical values that can be the output (y-values or f(x) values) of a function.

For example, consider the linear function \( f(x) = 2x + 1 \). What possible values can be put into this function so that an output can be found? In other words, what x-values will result in a corresponding y-value? What output values are possible?

Enter the function \( f(x) = 2x + 1 \) into your calculator. Graph the function in the standard window. A sketch of the graph is shown below in a smaller window. How does the graph help determine the possible input (domain) values? How does the graph help determine the possible output (range) values?

![Graph of the function f(x) = 2x + 1](image)

Create a table for the function \( f(x) = 2x + 1 \) on your calculator. How can the table help determine the possible input (domain) values? How does the table help determine the possible output (range) values?

When using the table to help determine the domain and range of a function, you must keep in mind that checking non-integer values of x may also be important.
Find the domain and range for each function below. First, think about what values make sense as inputs and as outputs. Then, enter the function rule into your calculator. Use the graph of the function and the corresponding table of values to help or to confirm your thinking.

1. \( f(x) = 7x - 2 \)
   - Domain: All real numbers
   - Range: All real numbers

2. \( g(x) = 2^x \)
   - Domain: All real numbers
   - Range: All real positive numbers

3. \( h(x) = 4 - 6x \)
   - Domain: All real numbers
   - Range: All real numbers

4. \( j(x) = 5(3)^x \)
   - Domain: All real numbers
   - Range: All real positive numbers

5. \( p(x) = |x - 3| \)
   - Domain: All real numbers
   - Range: All non-negative real numbers

6. \( y = 8 \)
   - Domain: All real numbers
   - Range: 8