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A Curriculum Study About Trigonometric Applications Aligned with the Common Core State Standards

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A Curriculum Study About Trigonometric Applications Aligned with the

Common Core State Standards

Timothy Wilson

The College at Brockport, State University of New York
A thesis submitted to the Department of Education and Human Development of the State University of New York College at Brockport in partial fulfillment of the requirements for the degree of Master of Education
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Abstract

This project illustrates a process of designing curriculum for the study of trigonometry, particularly applications of trigonometry involving the Law of Sines and the Law of Cosines in precalculus. Many of these topics were previously taught in Algebra II and Trigonometry under NCTM standards. This curriculum is scaffolded to give access to precalculus learners that are below grade level, while maintaining the level of rigor and reasoning requisite for success in a Calculus course. Thus, the curriculum is particularly suited for learners in traditionally underperforming urban districts.
Chapter One: Introduction

This curriculum project is designed with practicality in mind. The demanding pace and high density of topics in the Algebra II and Trigonometry curriculum makes it difficult for current teachers to transition from the National Council of Teachers of Mathematics (NCTM) standards to the Common Core State Standards (CCSS). In addition, current textbooks do not adequately address the CCSS: “textbooks…are not targeted toward the goal of applying quantitative methods and techniques to real world problems” (Green and Emerson, 2010). The purpose of this curriculum project is to provide teachers with materials and information so they can offer students learning experiences in the context of a situation. This curriculum was written with teachers in mind with plans matched to specific problem situations, solution techniques, and student conceptions with target objectives. This thesis will serve to examine trigonometry curriculum with special focus dedicated to modeling with mathematics.

With the implementation of the CCSS in Mathematics, there is a clear paradigm shift from the “mile wide, inch deep” NCTM curriculum to “fewer, clearer, higher” learning standards (CCSSI Considerations, 2013). The CCSS “intend to set forward thinking goals for student performance based in evidence about what is required for success…[The CCSS] must ensure all American students are prepared for the global economic workplace” (CCSSI Standards-Setting). In short, “the standards as a whole must be essential, rigorous, clear and specific, coherent, and internationally benchmarked” (CCSSI Standards-Setting, 2013).

High school teachers may be unfamiliar with the nuances of the CCSS learning standards, presenting new challenges to teaching Algebra II and precalculus curriculum. Thus, this thesis gives an opportunity to explore precalculus trigonometry curriculum in great detail, which may
help teachers more successfully grasp the fundamentals of the new standards and expedite the process of implementing the CCSS in their classroom instruction.

Unique to the CCSS, the entirety of mathematics instruction is to be administered in a way that develops eight standards of mathematical practice. These standards “describe varieties of expertise that mathematics educators at all levels should seek to develop in their students” (Standards for Mathematics Practice, 2016). There are five parts to this thesis. First, it provides a discussion about the shift from NCTM to CCSS standards and examines the scope and sequence of trigonometry pacing and instruction in the High School curriculum. Second, it investigates the meaning behind the Standards of Mathematics Practice (MP.4), mathematical modeling, and the implications of modeling for problem solving in mathematics. Third, it examines current research about trigonometry instruction and gatekeepers to student learning in general. Fourth, it discusses the Law of Sines and the Law of Cosines, deriving both as per CCSS standards, and provides educational practices to make these rigorous topics more easily accessible to high school students. Finally, it examines a current CCSS trigonometry sample task in order to discuss implications toward how the Law of Sines and the Law of Cosines is applied throughout the CCSS.

The ultimate goal of this project is to develop an effective unit plan for teaching trigonometric applications in oblique triangles. The unit design is aligned with CCSS, existing research on how students learn trigonometry, and employs techniques available to enhance student understanding. In designing this unit plan, the intended audience is teachers of mathematics. The plan will match specific problem situations, solution techniques, and student conceptions with target objectives.
Chapter Two: Literature Review

The Common Core State Standards Overview

The CCSS specifies eight standards for mathematical practice that apply through each course K-12, that coupled with the content standards allow students to “experience mathematics as a coherent, useful, and logical subject that makes use of their ability to make sense of problem situations” (National Governors Association, 2010).

The eight Standards for Mathematical Practice are as follows: (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (4) Model with mathematics; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure; (8) Look for and express regularity in repeated reasoning (National Governor’s Association for Best Practices; Council of Chief State School Officers, 2010, p. 5-7). The CCSS requires students to apply mathematics to model real world problems and to use the results to inform and justify their decisions. Applying mathematics in novel situations is expected in math related careers (National Governor’s Association, 2010).

Applications, or modeling, “take the form of problems to be worked on individually as well as classroom activities centered on application scenarios” (National Governor’s Association, 2010). A major goal of the high school standards is for students to practice applying mathematical ways of thinking to real-world issues, preparing students to think and reason mathematically (Rust, 2012). Though modeling is a standard of mathematical practice in all grades, note that in high school it is also a content category marked by a (*) symbol (National Governor’s Association, 2010). The CCSS focus more on depth or understanding rather than breadth of topic knowledge (Rust, 2012). Modeling requires students to use math and statistics
to analyze empirical situations and link classroom mathematics to everyday life, work and decision making (Rust, 2012). Mathematical modeling will be discussed at length later.

The necessity for fewer learning standards in favor of in depth problem solving opportunities is clear. See figure 1 for the projected CCSS Curriculum sequence.

Figure 1. Projected curriculum overview for CCSS

<table>
<thead>
<tr>
<th>Date</th>
<th>Grade 9 - Algebra I</th>
<th>Grade 10 - Geometry</th>
<th>Grade 11 - Algebra II</th>
<th>Grade 11 - Precalculus</th>
</tr>
</thead>
<tbody>
<tr>
<td>9/1/12</td>
<td>M1: Relationships Between Quantities and Reasoning with Equations (30 days)</td>
<td>M1: Congruence, Proof, and Constructions (45 days)</td>
<td>M1: Polynomial, Rational, and Radical Relationships (45 days)</td>
<td>20 days</td>
</tr>
<tr>
<td>10/1/12</td>
<td>M2: Descriptive Statistics (20 days)</td>
<td>M2: Similarity, Proof, and Trigonometry (45 days)</td>
<td>M2: Functions (45 days)</td>
<td>20 days</td>
</tr>
<tr>
<td>11/1/12</td>
<td>M3: Linear and Exponential Relationships (40 days)</td>
<td>M3: Functions (45 days)</td>
<td>M3: Functions (45 days)</td>
<td>20 days</td>
</tr>
<tr>
<td>12/1/12</td>
<td>State Examinations</td>
<td>State Examinations</td>
<td>State Examinations</td>
<td>20 days</td>
</tr>
<tr>
<td>1/1/13</td>
<td>State Examinations</td>
<td>State Examinations</td>
<td>State Examinations</td>
<td>20 days</td>
</tr>
<tr>
<td>2/1/13</td>
<td>M4: Expressions and Equations (30 days)</td>
<td>M4: Connecting Algebra and Geometry through Coordinates (15 days)</td>
<td>M4: Rational and Exponential Functions (25 days)</td>
<td>20 days</td>
</tr>
<tr>
<td>3/1/13</td>
<td>M5: Quadratic Functions (30 days)</td>
<td>M5: Inferences and Conclusions from Data (40 days)</td>
<td>M5: Probability and Statistics (30 days)</td>
<td>20 days</td>
</tr>
<tr>
<td>4/1/13</td>
<td>Review and Examinations</td>
<td>Review and Examinations</td>
<td>Review and Examinations</td>
<td>20 days</td>
</tr>
<tr>
<td>5/1/13</td>
<td>Review and Examinations</td>
<td>Review and Examinations</td>
<td>Review and Examinations</td>
<td>20 days</td>
</tr>
<tr>
<td>6/1/13</td>
<td>Review and Examinations</td>
<td>Review and Examinations</td>
<td>Review and Examinations</td>
<td>20 days</td>
</tr>
</tbody>
</table>

Note that dates approximations are based on a fall student day of 9/1/12 and last day of 6/26/13.

National Governors Association Center for Best Practices, 2010

Under NCTM New York State (NYS) Standards, students learn trig ratios in 9th grade, then the entirety of trigonometry within Algebra II and Trigonometry in grade 11 (National Council of Teachers of Mathematics, 2000). Some important distinctions between NYS NCTM
are noteworthy. Trig ratios are now taught in Geometry. Eleventh grade is reserved for trig graphs modeling periodic functions and proving trig identities. The majority of trigonometry is now taught in 12th grade precalculus: exact trig values; the unit circle; restricting the domain of a trig function to allow for its inverse to be a function; proving trig identities; deriving the area formula of a triangle and parallelogram using trigonometry; the Law of Sines and the Law of Cosines. Notably, there is no mention of double angle and half angle formulas (National Governors Association Center for Best Practices, 2010).

The Necessity of Mathematical Modeling

Standards for Mathematical Practice MP.4, Model with Mathematics states:
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationship using such tools as diagrams, two-way tables, graphs, flowcharts, and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical
results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

(National Governors Association, 2010)

Current textbooks do not sufficiently allow students to model mathematically (Green and Emerson, 2010). Textbooks “teach students particular mathematical techniques and procedures rather than to help students develop thinking skills necessary for analyzing the kinds of quantitative information they will encounter in their professional lives” (Green and Emerson, 2010, p.2).

Green and Emerson (2010) define knowledge as factual, conceptual, procedural, or metacognitive. Knowledge is cross-referenced against a cognitive process domain describing what students are doing with the knowledge: remembering, understanding, applying, analyzing, evaluating, or creating. Each cognitive process respectively is a higher order of thinking. Typically, textbooks only require students to remember and apply knowledge. However, problem solving in realistic settings tends to reach into the upper levels of processing: analyze, evaluate, and create. The difference is that true problem solving requires the problem solver to think reflectively, an exercise typically left untapped in most current textbooks (Emerson and Green, 2010).

Mathematical modeling is essentially a process by which the modeler begins with a real world situation by which the modeler constructs a model based on assumptions (Emerson and Green, 2010). Furthermore, modeling problems inevitably involve a “mathematizable situation, a mathematical object, a purpose or question that prompted the modeling activity, and the relationships between these things and the modeler” (Zbiek and Conner, 2006). Emerson and Green (2010) frame mathematical modeling as an
iterative process: creating the model involves abstraction, simplification, and quantification of real-world phenomena (Emerson and Green, 2010). The modeler analyzes the problem in the model world using mathematical tools, then transitions back to the real world by making meaning of the results from the model (Emerson and Green, 2010). This process is usually repeated iteratively, with each cycle informing the next (Emerson and Green, 2010).

Zbiek and Connor (2006) formalize this transition from a real world situation to a model setting as “mathematization.” The cyclic, iterative nature of modeling is illustrated in figure 2. Subprocesses of mathematical modeling are depicted with bidirectional arrows and subprocesses that have subprocesses are show with circular arrows.

Figure 2. Modeling Process diagram
Zbiek and Conner (2006) explain that each modeler’s work takes a path through the diagram, but that different modelers may take different pathways depending on what unique background knowledge, intuitions, and beliefs they possess. In other words, when faced with a mathematical modeling problem, there are many paths of inquiry students may take to reach the same conclusion. It is when students are forced to “mathematize” that the richest modeling experiences occur (Zbiek and Conner, 2006). This is because students must “choose the right mathematics” based on their unique knowledge and intuitions (Zbiek and Conner, 2006).

In fact, research indicates that solving real world problems involving mathematical modeling motivates students to engage in mathematics. Zbiek and Connor (2006) suggest that as students explore, they may become excited by the context and may become excited as they see associations between mathematics and some real world issue. In addition to providing motivation, mathematical modeling provides opportunities to learn mathematics (Zbiek and Connor, 2006).

One can draw many parallels between the mathematical modeling process and the Standards of mathematical practice. For instance, choosing the right mathematics is tied directly to MP.5: Use appropriate tools strategically. Mathematizing real-world situations is tied to MP.1 and MP.2: Make sense of problems and persevere in solving them, and reason abstractly and quantitatively, respectively. It then serves to conclude that through mathematical modeling, students are employing most of what the Standards for Mathematical Practice describe as a “mathematically proficient student” (National Governor’s Association, 2010).
Learning Trigonometry

Trigonometry education in the United States has been criticized as being a “house without a foundation” (qtd. in Van Sickle, 2011). Student understanding is “more mechanical than is ideal (Ginsberg et al., 2005). In addition, trigonometry is typically a topic “very few students like” that most students “hate and struggle with” (Gur, 2009). Effective implementation of curriculum may allow students to demystify the beauty and practicality of mathematics (Ginsberg et al., 2005). A conceptual understanding of trigonometry grants students access to understanding monumental concepts marked by historical human achievement such as celestial motion, periodic functions, and architectural construction. Trigonometric functions are a gateway to understanding increasingly dynamic and applicable mathematical models (Brummelin, 2010).

Gur (2009) found that many students possess an incomplete understanding of the three major ways to view sine and cosine: as coordinates of a point on the unit circle, as horizontal and vertical distances that are graphical entailments of those coordinates, and as ratios of sides of a reference triangle. In addition, “memorizing methods provide students knowledge of trigonometry only for a brief moment of time, but this knowledge is not retained by the students in the long run (Gur, 2009). In agreement, Van Sickle (2011) asserts “modern technology has shown geometric models to be extremely helpful for cultivating students’ understandings.

It is possible to examine the history of trigonometry education to inform future teaching practices (Van Sickle, 2011). Van Sickle (2011) postulates, “it would be helpful to include the line system as part of trigonometry education in the United States.” The line system she is referring to is the method of teaching trigonometry by defining trig functions as line segments within a circle (See figure 3).
The integration of modern technology, namely *Geometer’s Sketchpad* can help students understand the relationship between the line segment and its corresponding trig graph for values of theta between 0 and 360 degrees through animation (Van Sickle, 2011).

(Abramowitz & Stegun, 1964)

This addition to the curriculum would incorporate the line system, promote student understanding, and help to connect the origins of trigonometry with its modern day uses (Van Sickle, 2011).

Gur (2009) concludes that many misconceptions relate to a “procept” or the ability to think of mathematical operations and object. For example, students have difficulty understanding that sin(x) is both a function and a value (Gur, 2009). Vaninsky (2010) corroborates this notion, stating “students frequently skip arguments of trigonometric functions…they write sin instead of sin(x).”
The Law of Sines and Cosines

The Law of Sines and the Law of Cosines are “of paramount importance” in the field of trigonometry because they establish relationships satisfied by the sides and the three angles of any triangle (Skurnick and Jaradi, 2006). These laws are as follows: given a triangle $\triangle ABC$ with sides $a$, $b$, $c$ opposite angles $A$, $B$, and $C$ respectively, The Law of Sines is

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

and the Law of Cosines is $a^2 = b^2 + c^2 - 2bc \cos A$. The CCSS require that students prove the Law of Sines and the Law of Cosines and use them to solve problems (National Governors Association, 2010). Historic proof techniques will now be demonstrated.

The Law of Sines

Consider $\triangle ABC$ with sides $a$, $b$, $c$ opposite angles $A$, $B$, and $C$ respectively. Draw auxiliary line segment $h$, altitude to $\triangle ABC$, perpendicular to side $AC$.

Notice that $\sin A = \frac{h}{c}$ and that $\sin C = \frac{h}{a}$. Solving both equations for $h$ gives $c \sin A = h$ and $a \sin C = h$. Since both equations equal $h$, we may set them equal to each other, giving

$$c \sin A = a \sin C.$$  Dividing both sides of the equation by $ac$ then gives $$\frac{\sin A}{a} = \frac{\sin C}{c},$$ and without loss of generality, we have the Law of Sines, $$\frac{\sin A}{a} = \frac{\sin C}{c} = \frac{\sin B}{b}.$$
The Law of Cosines

Consider \( \triangle ABC \) with sides \( a, b, c \) opposite angles A, B, and C respectively. Draw altitude \( \overline{BD} \) from angle B, perpendicular to side \( \overline{AC} \). Notice that \( \overline{BD} = a \sin C \), \( \overline{CD} = a \cos C \), and \( \overline{DA} = b - a \cos C \). By the Pythagorean Theorem, it follows that

\[
c^2 = (a \sin C)^2 + (b - a \cos C)^2
\]

\[
c^2 = a^2 \sin^2 C + b^2 - 2ab \cos C + a^2 \cos^2 C
\]

\[
c^2 = a^2 \sin^2 C + a^2 \cos^2 C + b^2 - 2ab \cos C
\]

\[
c^2 = a^2 (\sin^2 C + \cos^2 C) + b^2 - 2ab \cos C
\]

\[
c^2 = a^2 + b^2 - 2ab \cos C
\]
Chapter Three: Unit Plan

The precalculus unit plan detailed below addresses trigonometric applications, namely those involving the Law of Sines, Law of Cosines, and the Area formula. This topic, while not wholly new to the Common Core State Standards (CCSS), adds that students must prove each formula. The precalculus unit Trigonometry Applications is aligned to the following standards.

HSG.SRT.D.10 (+) – Prove the Laws of Sines and Cosines and use them to solve problems.

HSG.SRT.D.11 (+) – Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g. surveying problems, resultant forces).

HSG.SRT.D.9 (+) – Derive the formula \( A = \frac{1}{2} ab \sin(C) \) for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

The lesson plans use the Understanding by Design UbD lesson planning format, adopted from Grant Wiggins and Jay McTighe (Wiggins & McTighe, 2005). Stage one of the UbD format is clarifying desired results. Educators identify what the student will know at the end of the lesson. This section also identifies what skills and vocabulary students will know to enable them to reach the desired results. The second stage is “assessment evidence,” where teachers communicate how what the students are doing will be assessed. This can include independent practice, group work, homework, checks for understanding, and ticket-out-the-doors. Stage three is the learning plan. The learning plan details activities as either acquisition, meaning-making, or transfer throughout the lesson. The UbD lesson design template is used at the school in which the author is employed, and as such, was selected for its use in this thesis.
Each lesson is designed to take one 72 minute block. This time frame is merely a suggestion. The teacher using the curriculum may decide to adjust the time allotted to each lesson to better meet the needs of his or her students. In addition to notes, ticket-out-the-doors (TOTDs) and suggested homework assignments are also located at the end of each lesson. Review materials are also included which could be used as a homework assignment, or as an in class work day Lesson 5. A unit test is also included which is designed to take roughly 45 minutes to an hour of class time. This embeds additional review time prior to the beginning of the examination on the day of the assessment. Although the main source of content is original content from the author, additional resources include emathinstruction, Core Plus Mathematics, and Illuminations. An answer key to all materials is included after the blank unit materials. Additional implementation strategies and suggestions are detailed extensively in the next chapter.
**Stage 1 – Desired Results**

<table>
<thead>
<tr>
<th>External Standard(s):</th>
<th>Essential Question: What is the Law of Sines, and how can it be used to find side lengths in a triangle?</th>
</tr>
</thead>
<tbody>
<tr>
<td>G-SRT.10: “Prove the Laws of Sines and Cosines and use them to solve problems” as a (+). The (+) indicates additional mathematics for all college and career-ready students.</td>
<td></td>
</tr>
</tbody>
</table>
| Long term goal | In the long run, students will independently….
- Derive the Law of Sines
- Use the Law of Sines to solve problems involving indirect measurement |
| Understanding goal | Students will understand that the Law of Sines can be used to solve for a missing side or angle of any triangle, right or non-right |
| Kid version | I can solve for a missing side of a triangle using the Law of Sines |
| Skills and Content (acquisition goal): | Use the Law of Sines to determine a missing side of a triangle |

**Stage 2 – Evidence of learning**

<table>
<thead>
<tr>
<th>Performance Tasks</th>
<th>Other Evidence (TOTD’s, quizzes)</th>
</tr>
</thead>
</table>
| - Examples 2, 3, and 5-8 | - Individual practice
- Practice with partners / small groups
- TOTD
- HW |

**Stage 3 – Deliberate Practice**

<table>
<thead>
<tr>
<th>Acquisition Activities:</th>
<th>Meaning-Making Activities</th>
<th>Transfer Activities:</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Determining the length of a side</td>
<td>- Explaining why steps are correct or incorrect</td>
<td>- Comparing methods to derive the Law of Sines</td>
</tr>
</tbody>
</table>
Recall: SOH CAH TOA

What kinds of triangles have a Hyp.?

Real life rarely has right triangles. We are going to explore non-right triangles using trigonometry.

Sketch triangle ABC and triangle PRQ and label each angle and side.

KNOW THIS:

- The largest side of a triangle is across from _________________.
- The smallest side is across from ____________________.

**NON-right triangle problems involving a missing side or angle may use either...**

<table>
<thead>
<tr>
<th>Law of Sines</th>
<th>OR</th>
<th>Law of Cosines</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} )</td>
<td>( a^2 = b^2 + c^2 - 2bc \cos A )</td>
<td></td>
</tr>
</tbody>
</table>

Today, we will derive the Law of Sines and use it to solve problems. Consider the following problem.
Ex1 Suppose that two park rangers who are in towers 10 miles apart in a national forest spot a fire that is far away from both of them. Suppose that one ranger recognizes the fire location and knows it is about 4.9 miles from that tower.

With this information and the angles given in the diagram below, the rangers can calculate the distance of the fire from the other tower.

One way to start working on this problem is to divide the obtuse triangle into two right triangles as shown below.

One class of students solved for the length of $\overline{AC}$ using the steps below.

( TPS) To the right of each step, give reasons for why or why not it is correct.

(1) $\frac{h}{b} = \sin 29^\circ$

(2) $h = b \sin 29^\circ$

(3) $\frac{h}{4.9} = \sin 53^\circ$

(4) $h = 4.9 \sin 53^\circ$

(5) $b \sin 29^\circ = 4.9 \sin 53^\circ$

(6) $b = \frac{4.9 \sin 53^\circ}{\sin 29^\circ}$

(Hirsch, C. R., 2015)
The approach for solving the fire problem (Ex1) on the previous page is almost identical to how the Law of Sines is derived.

**Ex2** (In pairs) Explain why each step in the following derivation is correct for $\triangle ABC$ below

\[
\frac{h}{b} = \sin A \\
h = b \sin A \\
\frac{h}{a} = \sin B \\
h = a \sin B \\
b \sin A = a \sin B \\
\frac{\sin A}{a} = \frac{\sin B}{b}
\]

**Ex3** *(TPS)* Modify the above derivation to show that $\frac{\sin B}{b} = \frac{\sin C}{c}$

The relationships between sides and angles of triangles shown in Ex2 and Ex3 is called the **Law of Sines**. It can be written in two equivalent forms.

<table>
<thead>
<tr>
<th>The Law of Sines</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ and also $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$</td>
</tr>
</tbody>
</table>

An important note: when solving problems, you only use ______ of the ratios at a time.
Ex4  (Whole Group) Suppose the park rangers spot a forest fire as indicated in the diagram below. Find the distances from each tower to the fire.

Ex5  (In pairs) In $\triangle ABC$, $m \angle B = 44^\circ$, $m \angle C = 53^\circ$, and $AC = 7$. Find the measure of side $AB$.

Ex6  (Individually) Using the same triangle as in Ex5, find the measure of side $BC$. 

(Hirsch, C. R., 2015)
Ex7

Find the perimeter of $\triangle DEF$.

Ex8

Two angles of a triangle measure $40^\circ$ and $56^\circ$. If the longest side of the triangle is 28cm, find the length of the shortest side to the nearest tenth of a centimeter.
Trig Applications Day 1 TOTD

Two angles of a triangle measure 50° and 75°. If the shortest side of the triangle is 28 cm, find the length of the longest side to the nearest tenth of a centimeter.
Trig Applications Day 1 HOMEWORK

1) Triangle SEA is shown below. Find the measure of $x$, the distance from A to S.

![Diagram of triangle SEA with angles and distances] (32 miles, 65° E, 75°)

2) In a triangle, two angles measure 14° and 110°. The side opposite the 14° angle measures 9.4 units. Find the length of the longest side.

3) Triangle ABC is shown below.

   ![Diagram of triangle ABC with sides labeled a, b, c] (Hirsch, C. R., 2015)

   a) If you were tasked to find the length of $BC$ using the Law of Sines, what is the minimal amount of information you would need to be given?

   b) Using your answer to part a), explain how you would use the given information to find the length of side $BC$. 

(Hirsch, C. R., 2015)
## Stage 1 – Desired Results

**External Standard(s):**
G-SRT.10: “Prove the Laws of Sines and Cosines and use them to solve problems” as a (+). The (+) indicates additional mathematics for all college and career-ready students.

**Essential Question:** When is the Law of Sines ambiguous and how do you determine the number of possible triangles possible in a problem situation?

**Long term goal** In the long run, students will independently....
- Use the Law of Sines to determine missing angle measures
- Determine whether zero, one, or two triangles are possible when the lengths of two sides and the measure of an angle not included between these sides are known

**Understanding goal**
Students will understand that the Law of Sines can be ambiguous when solving for a missing angle and sometimes 0, 1, or 2 triangle(s) exist that satisfy a problem situation.

**Kid version:**
I can determine missing angles and the number of triangles possible using the Law of Sines

**Skills and Content (acquisition goal):**
Use the Law of Sines to determine the possible missing angles in a triangle
Determine the number of possible triangles using the Law of Sines

## Stage 2 – Evidence of learning

### Performance Tasks
- Notes

### Other Evidence *(TOTD’s, quizzes)*
- Individual practice
- Practice with partners / small groups
- TOTD
- HW

## Stage 3 – Deliberate Practice

### Acquisition Activities:
Practice problems (Part II)

### Meaning-Making Activities
- Exploring SSA with a compass

### Transfer Activities:
- Using the Law of Sines to determine missing angle measures, testing to see if angle measures are reasonable
I. Exploring the SSA ambiguous case

Consider the following example...

In \( \triangle ABC \), \( a = 10 \), \( b = 16 \) and \( m\angle A = 150^\circ \). Find \( \sin B \).

What if the question above asked you to find \( m\angle B \) to the nearest degree?

Is this angle measure possible? Explain.

The Law of Sines is **ambiguous** (may have different answers) when solving for angles. This occurs when given two sides and the *non*-included angle of a triangle (SSA). There are either 0, 1, or 2 possible triangles.

To show this, we will fix non-included angle \( A \), and sides \( b \) and \( a \).

( TPS): If we fix these three parts, which parts are you allowed to change?

The altitude \( h \) is also labeled. Use a compass to explore each case.

**Case 1**: The third side is smaller than the altitude \( (a < h) \)
II. Practice Problems

**Exercise #1:** In triangle $ABC$, which is shown below but not drawn to scale, it is known that $AC = 6$, $AB = 10$, and $m\angle B = 36^\circ$. Determine *all possible* values for $m\angle C$ to the nearest tenth.
CURRICULUM STUDY: TRIGONOMETRIC APPLICATIONS

Exercise #2: In triangle $ABC$, which is shown below but not drawn to scale, it is known that $AC = 10$, $AB = 14$, and $m \angle C = 80^\circ$.

(a) Solve an equation to find all possible values for the measure of $B$ to the nearest tenth.

(b) Considering the measures the angles of any triangle must sum to be $180^\circ$, why must we reject the obtuse solution from part (a)?

Exercise #3: Explain why the triangle shown below cannot exist by finding all possible values for $m \angle C$.

Exercise #4: In $\triangle DEF$, $m \angle E = 72^\circ$, $DE = 12$, and $DF = 15$. How many triangles are possible given this information?

(1) 1
(2) 2
(3) 3
(4) 0
Trig Applications Day 2 TOTD

Given $\triangle ABC$ such that $A = 30^\circ$, $a = 20$, and $b = 16$, determine the number of possible triangles.
Trig Applications Day 2 HOMEWORK

SKILLS

Diagrams given in problems are not drawn to scale and angles that appear acute may in fact be obtuse and vice versa.

1. In \( \triangle CDE \) shown below \( CE = 10, \ CD = 8, \) and \( m\angle E = 40^\circ \). How many possible values exist for \( m\angle D \)?
   - (1) 1
   - (2) 2
   - (3) 3
   - (4) 0

2. In \( \triangle ABC \) shown \( m\angle C = 25^\circ, \ c = 15, \) and \( b = 20 \). Angle \( B \) is
   - (1) acute only
   - (2) right only
   - (3) obtuse only
   - (4) acute or obtuse

3. For which value of \( f \) shown below will there be no solution for angle \( E \) in triangle \( DEF \)?
   - (1) 9
   - (2) 10
   - (3) 11
   - (4) 5

4. Accurate to the nearest tenth the largest possible value of \( m\angle N \) in the diagram below is
   - (1) 49.8°
   - (2) 130.2°
   - (3) 37.3°
   - (4) 105.6°

5. How many triangles can be formed in which the shortest side measures 9 inches, the longest side measures 14 inches and the smallest angle measures 48°?
   - (1) 1
   - (2) 2
   - (3) 3
   - (4) 0

(eMathinstruction, 2016)
## Stage 1 – Desired Results

### External Standard(s):
- HS.G.SRT.D.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.
- HS.G.SRT.D.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

### Essential Question:
What is the Law of Cosines and how can it be used to find side lengths or angle measures in triangles?

### Long term goal
In the long run, students will independently:
- Derive the Law of Cosines
- Use the Law of Cosines to determine missing angle or side measures
- Decide on the appropriate approach to solve a triangle problem

### Understanding goal
Students will understand that the Law of Cosines is a useful tool to find a missing side or angle when a triangle problem involves three sides and an angle.

**Kid version:**
I can decide when to use the Law of Cosines
I can use the Law of Cosines to find a missing side or angle in a triangle

### Skills and Content (acquisition goal):

## Stage 2 – Evidence of learning

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## Stage 3 – Deliberate Practice

### Acquisition Activities:
- Part I: getting comfortable with a new formula
- Part II: solving problems

### Meaning-Making Activities:
- Part I: Deriving the Law of Cosines

### Transfer Activities:
Using the Law of Cosines to solve for missing sides or angles of triangles
The Satellite dish problem
DAY 3: THE LAW OF COSINES

In the past two lessons, we used the Law of Sines to determine a missing side or angle. In each situation, we involved two sides and two angles where one side or angle was missing. For problems with different combinations of angle measures and side lengths being known, a second property is helpful: The Law of Cosines.

I. Getting comfortable with a new formula

The Law of Cosines states that in any triangle $ABC$ with sides of length $a$, $b$, and $c$,

$$c^2 = a^2 + b^2 - 2ab \cos C$$

1. If you are solving for a missing side, what number of side(s) and angle(s) measurements must be known?

2. If you are solving for a missing angle, what number of side(s) and angle(s) measurements must be known?

When we are solving for an angle or a side, always rearrange the formula to include the missing side or angle’s letter first. This means we could use the above formula to solve for either side $c$ or angle $C$.

3. Rearrange the letters in the formula to set up to solve for side $a$ or angle $A$.

4. Rearrange the letters in the formula to set up to solve for the side $b$ or angle $B$.

5. If you were given triangle $XYZ$, set up the formula to solve for missing side $y$ or angle $Y$. 
II. Deriving the Law of Cosines

Derivations of the Law of Cosines depend on whether the angle under consideration is acute or obtuse. In our first setup, \( \triangle ABC \) contains obtuse \( \angle B \) and \( h \) is the altitude of the triangle drawn from vertex \( C \).

1. Explain why each step in the following derivation is true. The reason for (5) is given.

(1) In right triangle BCE,
\[
\begin{align*}
    x^2 + h^2 &= a^2 \\
    h^2 &= a^2 - x^2
\end{align*}
\]

(2) In right triangle ACE,
\[
\begin{align*}
    h^2 + (x + c)^2 &= b^2 \\
    h^2 &= b^2 - (x + c)^2
\end{align*}
\]

(3) \( a^2 - x^2 = b^2 - (x + c)^2 \)

(4) \( b^2 = a^2 + c^2 + 2cx \)

(5) In right triangle BCE,
\[
\begin{align*}
    \cos(m\angle CBE) &= \frac{x}{a} \\
    x &= a \cos(m\angle CBE) = -a \cos B
\end{align*}
\]
Reason: Since angle B is obtuse but less than 180°, the cosine of angle B is negative.

(6) \( b^2 = a^2 + c^2 - 2ac \cos B \)

(Hirsch, C. R., 2015)
For the case with an acute angle, you will be guided through the proof. Complete this in groups of 3. Discuss your work as you go.

Begin with $\triangle ABC$ with altitude $k$ drawn from vertex $C$.

1. The altitude separates $\triangle ABC$ into two right triangles. Use the Pythagorean theorem to write two equations, one relating $k$, $b$, and $c - x$, and another relating $a$, $k$, and $x$.

2. Notice that both equations contain $k^2$. Solve each equation for $k^2$.

3. Since both of the equations in Question 2 are equal to $k^2$, they can be set equal to each other. Set the equations equal to each other to form a new equation.

4. Notice that the equation in Question 3 involves $x$. However, $x$ is not a side of $\triangle ABC$. We will try to rewrite the equation in Question 3 so that it does not include $x$. Begin by expanding $(c - x)^2$. 
5. Solve the equation in Question 4 for $b^2$.

6. The equation in Question 5 still involves $x$. To eliminate $x$ from the equation, we will attempt to substitute an equivalent expression for $x$. Write an equation involving both $\cos B$ and $x$.

7. Solve the equation from Question 6 for $x$.

8. Substitute the equivalent expression for $x$ into the equation from Question 5. The resulting equation contains only sides and angles from $\triangle ABC$. This equation is the Law of Cosines.

(Illuminations, 2016)
III. Using the Law of Cosines to Solve Problems

Exercise #1: If $m\angle C = 90^\circ$ then the first form of the Law of Cosines results in what famous theorem? Justify your answer by substituting $90^\circ$ for $C$.

Exercise #2: In $\triangle ABC$ below, $AB = 8, BC = 13$, and $m\angle B = 110^\circ$. Find the length of $\overline{AC}$ to the nearest tenth.

Exercise #3: In triangle $DEF$ below, $DE = 11, EF = 15$, and $DF = 20$. Determine $m\angle D$ to the nearest degree.

Exercise #4: A triangle has three sides that measure 8, 14, and 20 centimeters. Find the measure of the largest angle of this triangle to the nearest tenth of a degree.
Trig Applications Day 3 TOTD

1) Given $\triangle XYZ$, make a sketch and label all angles and sides. Then, write the correct setup to solve for angle $Z$ using the Law of Cosines.

2) If angle $Y$, angle $X$, and sides $x$ and $y$ are known, explain whether or not the Law of Cosines could be used to solve for angle $Z$.

3) Find $BC$. 

![Diagram](image_url)
SKILLS

1. In each of the triangles below, a S.A.S. scenario is given. Find the value of $x$ in each case to the nearest tenth using the Law of Cosines.

   (a)
   \[
   \begin{array}{c}
   \text{10} \\
   \hline
   \text{16} \\
   \text{48°} \\
   \end{array}
   \]

   (b)
   \[
   \begin{array}{c}
   \text{12} \\
   \hline
   \text{x} \\
   \text{7} \\
   \end{array}
   \]

2. In each of the triangles below, a S.S.S. scenario is given. Using the Law of Cosines, find the measure of $\theta$ as marked in each triangle accurate to the nearest degree.

   (a)
   \[
   \begin{array}{c}
   \text{18} \\
   \hline
   \text{16} \\
   \text{27} \\
   \end{array}
   \]

   (b)
   \[
   \begin{array}{c}
   \text{34} \\
   \hline
   \text{18} \\
   \text{20} \\
   \end{array}
   \]

3. In a triangle whose side lengths measure 20, 21, and 29 inches respectively, the largest angle is

   (1) acute
   (2) right
   (3) obtuse
   (4) either acute or obtuse

(eMathinstruction, 2016)
### Stage 1 – Desired Results

<table>
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<tr>
<th>External Standard(s):</th>
<th>Essential Question: How do you use trigonometry to determine the area of a triangle when the height is not known?</th>
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<tbody>
<tr>
<td>HSG.SRT.D.9 (+) Derive the formula $A = \frac{1}{2}ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.</td>
<td>Long term goal In the long run, students will independently....</td>
</tr>
<tr>
<td></td>
<td>- Derive and apply the area formula $\frac{1}{2}ab \sin C$ for a triangle</td>
</tr>
</tbody>
</table>

**Understanding goal:** Students will understand that area is a function of two side lengths and the measure of their included angle and apply the area formula to determine area in a triangle given two side lengths and the measure of their included angle.

**Kid version:** I can derive the area formula $A = \frac{1}{2}ab \sin C$. I can use the area formula $A = \frac{1}{2}ab \sin C$ to determine the area of a triangle when the height is not known.

**Skills and Content (acquisition goal):**

- Determine area using the area formula

### Stage 2 – Evidence of learning

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### Stage 3 – Deliberate Practice

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DAY 4: FINDING AREA USING TRIGONOMETRY

In the first three lessons, we developed techniques and formulas for finding missing sides or angles of triangles. It is often useful to know the exact amount of space a figure contains. We will develop a more sophisticated area formula for determining the area of a triangle. Let’s begin with the formula you are already familiar with.

Recall:

Area of a Triangle =

Ex

In the above example, we used the fact that a right triangle’s height is always known. What if we want to find the area of a triangle when the height is not known? With a partner, follow each step to arrive at the new formula.

1. Draw altitude from vertex A to side BC and label its length \( h \). Notice that \( h \) is the height of \( \triangle ABC \).
2. Use your familiar triangle area formula to write the area of this triangle using \( a \) and \( h \).
3. Since \( h \) is not the length of a side of \( \triangle ABC \), we must work to represent \( h \) using relationships between given sides and angles. To do this, write an equation for \( \sin C \) relating \( h \) and side \( b \).
4. Solve this equation for \( h \).
5. Substitute the expression from part 4 into your area formula in part 2. Verify that you have the same formula as another group and this this matches the formula on the next page.
II. Getting comfortable using our new formula.

\[ \text{Area} = \frac{1}{2} ab \sin C \]

The formula you derived is able to find the area of a triangle if sides a, and b, and angle C are known.

1. What is important to note about how angle C is positioned in relation to sides a and b?

2. If given side measures a and c, what angle would need to be given in order to apply the area formula? Modify the above area formula to show this.

3. If given angle A, what two sides would also need to be given in order to apply the area formula? Modify the above area formula to show this.

4. Modify the area formula to represent the area of a different triangle, \( \triangle ACE \). Compare your response to another group’s. What did you notice? Together, determine at least one additional area formula for \( \triangle ACE \).

5. Is it possible to use the area formula to solve for a missing side or angle of a triangle? Explain your thinking.
III. Using the area formula to solve problems

Ex Find the area of $\triangle DEF$ if $DE = 14$, $EF = 9$, and $\angle E = 30^\circ$.

Ex The adjacent sides of parallelogram $ABCD$ measure 12 and 15. The measure of the obtuse angle of the parallelogram is $135^\circ$. Find the area of the parallelogram to the nearest tenth.

Ex Mr. Wilson is trying to determine the amount of paint he will need for the triangular portion in front of his house. A gallon of paint he will use will cover 150 square feet. If the portion he must cover has the shape of an isosceles triangle, shown below, with legs 36 feet and base angles of $30^\circ$, determine the minimum number of gallons of paint Mr. Wilson will need. Note that Mr. Wilson can only buy full gallons of paint.

(eMathinstruction, 2016)
Trig Applications Day 4 TOTD

1) In general, what information needs to be given in order to determine the area of a triangle using the formula $Area = \frac{1}{2}ab\sin C$?

2) Determine the area of the following triangle.

3) Create your own real world problem involving finding the area of a triangle using the formula from today's lesson and solve it.
1. Modify the diagram, wording, and work you did in the first part of today’s lesson to derive the area formula \( \text{Area} = \frac{1}{2}bc\sin A \).

2. Find the area of the triangle with the given measurements to the nearest tenth.
   a. In \( \triangle ABC \), \( b = 14.6 \), \( c = 12.8 \), \( m\angle A = 56^\circ \)

   b. In \( \triangle PQR \), \( p = 212 \), \( q = 287 \), \( m\angle R = 124^\circ \)

   c. In \( \triangle ABC \), \( b = 7 \), \( c = 8 \), \( \sin A = \frac{3}{5} \)
3. Find the area of a parallelogram if the measures of two adjacent sides are 40 feet and 24 feet and the measures of one angle of the parallelogram is 30 degrees.

4. The following diagram shows the peak of a roof that is in the shape of an isosceles triangle. A base angle of the triangle measures $50^\circ$ and each side of the roof is 20.4 feet. Determine to the nearest tenth of a square foot the area of the triangular region.
1. The area of an equilateral triangle, in simplest radical form, whose side lengths measure 10 is

   (1) $50\sqrt{2}$  
   (2) $22\sqrt{7}$  
   (3) $25\sqrt{3}$  
   (4) $100\sqrt{2}$

2. Vivian would like to create a window in the shape of an equilateral triangle whose area is 100 square inches. To the nearest tenth of an inch, how long should each side of this triangle be?

   (1) 17.6  
   (2) 9.7  
   (3) 15.2  
   (4) 22.8

3. In $\triangle ABC$, $a = 4, b = 2$ and $\sin A = \frac{6}{7}$. Which of the following must be the value of $\sin B$?

   (1) $\frac{3}{7}$  
   (2) $\frac{1}{2}$  
   (3) $\frac{2}{3}$  
   (4) $\frac{12}{7}$

4. Rohan takes measurements of triangle $ABC$ and finds that $AC = 5$, $AB = 8$ and $m\angle C = 46^\circ$. How many triangles exist that could have the measurements that Rohan took?

   (1) 1  
   (2) 2  
   (3) 3  
   (4) 0

(eMathinstructor, 2016)
CURRICULUM STUDY: TRIGONOMETRIC APPLICATIONS

5. Given the diagram below, then the value of $x$, to the nearest tenth, is

\[
\begin{array}{cc}
\text{(1) 32.8} & \text{(3) 30.1} \\
\text{(2) 13.4} & \text{(4) 18.7}
\end{array}
\]

6. Marissa would like to create a flower patch in the shape of an isosceles triangle. She has enough room so that the triangle has dimensions as shown below. Determine the area of Marissa’s triangle to the nearest square foot.

7. Explain why the triangle shown below cannot exist.

(eMathinstruction, 2016)
A triangle has side lengths of 8, 13, and 18 inches. Determine the smallest angle of this triangle to the nearest degree.

In the diagram shown below, points D, E, and F are collinear. If $m \angle D = 22^\circ$, $m \angle CEF = 52^\circ$ and $DE = 2$: then find the length of $CF$ to the nearest tenth.

Jonathan has created a triangular room in his house that has side lengths of 12 feet, 16 feet and 20 feet. He would like to cover it with flooring that costs $3.25 per square foot. Assuming there is no wasted flooring, calculate, to the nearest dollar, how much Jonathan will need to spend on flooring.
Trig Applications Unit Test

Part I – Multiple Choice

Each question is worth 2 points. No partial credit will be awarded.

1. In \( \triangle ABC \), \( AB = 12 \), \( BC = 16 \) and \( m\angle B = 72^\circ \). Which of the following represents the area of \( ABC \) to the nearest integer?

   (1) 91
   (2) 96
   (3) 78
   (4) 183


2. An isosceles triangle has legs with lengths of 24 inches and base angles that measure 56°. To the nearest square inch, which of the following is the area of this isosceles triangle?

   (1) 184
   (2) 267
   (3) 239
   (4) 108

3. A triangle whose area is 15 has two sides of length 6 and 10. If the angle created by these two sides is obtuse, then its measure must be

   (1) 120°
   (2) 135°
   (3) 30°
   (4) 150°
4 In $\triangle DEF$, $EF = 18$, $m\angle D = 75^\circ$ and $m\angle F = 20^\circ$. To the nearest tenth, the length of $DE$ is

(1) 4.8  
(2) 5.0  
(3) 6.4  
(4) 7.8

5 In $\triangle MNP$, $m\angle N = 30^\circ$, $MN = 10$ and $MP = 6$. Given this information angle $P$ could be

(1) acute only  
(2) right only  
(3) acute or obtuse  
(4) obtuse only

6 In the triangle shown below, what is the measure of the smallest angle, to the nearest degree?

(1) 29°  
(2) 47°  
(3) 76°  
(4) 22°

(eMathinstruction, 2016)
CURRICULUM STUDY: TRIGONOMETRIC APPLICATIONS

Part II – Extended Response

Each question is worth 2 points. Partial credit may be awarded.

7. In $\triangle ABC$ shown below, it is known that angle $B$ is obtuse. Find the measure of angle $B$ to the nearest degree.

8. A radar is tracking the location of a plane. Initially the plane is at a distance of 6.5 miles from the radar. After the radar rotates through an angle of 115$^\circ$, the plane is now at a distance of 3.2 miles from the radar. What straight line distance, to the nearest tenth of a mile, has the plane flown between these two readings?
Part III – Extended Response

Each question is worth 4 points. Partial credit may be awarded.

9. In \(\triangle ABC\) shown below, not drawn to scale, it is given that \(\angle C = 44^\circ\), \(AC = 10\) and \(AB = 8\). Find all possible areas of \(\triangle ABC\) to the nearest integer.

10. In quadrilateral \(ABCD\) it is known that \(AB = 16\), \(BC = 24\), \(CD = 36\), \(\angle C = 32^\circ\) and \(\angle ABD = 64^\circ\). Determine, to the nearest tenth the length of \(AD\).

(eMathinstruction, 2016)
Chapter 4: Conclusion

This curriculum has been developed with teachers in mind, by a teacher in the field, with the intention that it is useful to current teachers in their quest to best help their students in meeting the rigorous standards set forth by the Common Core. This unit plan may be used as a resource in the classroom to teach students to better understand trigonometry. The author has implemented this curriculum in his precalculus class. The following is a list of suggestions and implementation recommendations based on the author’s experience.

Day 1 – The Law of Sines

- When discussing the relationship between angles and sides on p.2, use the analogy of opening a door. The larger the angle the door makes with the wall, the larger the opening through the doorway becomes.
- Resist presenting the equivalent forms of the Law of Sines and Cosines when the formulas are first presented on p.2. Opportunities to do this will occur later in the curriculum.
- On p.3, discuss why the sides of the triangle are labeled as such. Students may struggle to understand x and 10 – x at first.
- Opportunity to differentiate: some students may struggle to write reasons initially on p.4. Tell students not all steps are showing between the ones that are shown and that it may be helpful for them to “do the math” and write the mathematical steps.
- If the last reason (divide by ab) is unclear, encourage groups to write it as multiple reasons such as “First, divide both sides of the equation by a. Next…”
• In Ex3, encourage the whole class to redraw the above diagram but to decide how to label the vertices correctly. After giving students time to struggle, you may want to differentiate by giving the correct diagram and labels to lower level students.

• On p.5, after viewing the Law of Sines formula, a simple check for understanding may be to have students set up the Law of Sines on white boards using 3 different letters they choose and display their equivalent Law of Sines to the entire class. Ensure that every student agrees that the whiteboards are all equivalent to one another.

• For Ex4, the group discussion should establish that the vertices can be labeled using any letters as long as the opposite sides are labeled correctly. You may need to show your class how to set their calculators to degree mode. You should also discuss the importance of rounding in this context. Since you are talking about miles from a fire, students may reason that the tenths place is sufficient.

• You must emphasize that if students are substituting values of sides that they have rounded, this will alter the result. Students must “arrow up” and use the unrounded values from the calculator memory.

• In Ex5, once pairs have reached a stopping point, have a whole group discussion and bring out differences in solution techniques: setups, labeling, etc. and establish that each different approach produced an equivalent result.
  o Access to a document camera can greatly aid this process.
  o You may also select a group that was unable to reach the correct result and analyze next steps and/or errors
Day2 – Law of Sines & Missing Angles: The Ambiguous Case

- On p.11, have a discussion in order for students to recall that finding an angle given the sine of an angle involves the use of the inverse sine function on the calculator.

Encourage students to avoid saying things like “sine negative 1” or “second sine,” and instead saying “the angle whose sine is…”

- Discuss the meaning of a fixed angle or side in a mathematical context as a whole group. Groups should understand they are allowed to change everything except the fixed parts.

- Groups may struggle to see the different possible triangles. Have different colored highlighters or markers available for students to trace each possible triangle.

- You may wish to do Exercise #1 as a whole group. The group discussion should establish that since sine is positive in the first and second quadrant and that these quadrants include angles from 0 to 180, there are two possible angles that have an equivalent sine measure.

  o Note, addressing the angle span will help students understand why the Law of Cosines is not ambiguous in later lessons.

- Discuss how to choose which sides and angles to use in a Law of Sines problem.

  Students should understand to set up a Law of Sines with one ratio the unknown angle whose opposite side is known.

- Exercise #3 may surprise students when their calculator returns an error message for the inverse sine of 1.18. Tell students to note it and move on.

  o During the whole group discussion following the group work time, establish as a group that the definition of sine is the opposite divided by the hypotenuse. The hypotenuse is always the largest side, so the sine of an angle can never be larger
than 1. When this occurs, you must have been given a triangle like the one in case 1 from the exploration activity.

- On the HW, some students may reason to solve for \( \sin F \) and get \( \frac{\sin 42^\circ}{7} \). The correct solution is the one that gives a value greater than one when substituted.

**Day3 – The Law of Cosines**

- On p.17, make sure students understand that they may be given more information, but that the answers to 1 and 2 are the minimum information required to use the formula.
- On p.17, have students take note that the same letter will appear at the beginning and end of the formula. This will help them answer 3 and 4.
- Please note that the derivation of the Law of Cosines is a standard, but only top students will understand its derivation. Before proceeding with p.18, tell your students that it is okay if they find the derivation confusing, that many people don’t fully understand it until they study it in college, and that there is still value in being exposed to it at this level. It is recommended by the author that you approach this as a whole group.

Students should be comfortable with the reasoning for steps 1, 2, and 3. The algebraic manipulation thereafter may confuse all but your top learners, particularly the reason at the end of step 5. There is value to students being exposed to this algebraic derivation, but do not feel the need to spend excessive amounts of time on this page. Ensure students that they are only responsible for using the formula at this level.

- On p.19, for #4, encourage weaker students to expand \((c - x)(c - x)\) off to the side, then to substitute it back into the problem using \((\ )’s
Students may ask how far they need to expand and simplify the equation. Encourage them to decide on their own because it will become evident in later steps.

- For #6, if students are stuck, remind them to refer back to the diagram. If they are still unsure, encourage them to cover up the left triangle.

- On p.21, Exercise #1 is a good opportunity to notice that $2abc\cos C$ is all one term composed of 4 factors. This is intended to help prevent the common error of combining $2ab$ with $a^2$ or $b^2$ when values are substituted for $a$ and $b$.

- For Exercise 2 and 3, let students work in groups

- After student work time, have a whole group discussion and establish that when solving for a missing side, the entire right hand side of the equation can be entered into the calculator. Also remind students that they must take the square root to get their answer.

- For Exercise 3, there will be many errors after students initially substitute. As a whole group, establish that the right hand side is comprised of 3 terms: $20^2$, $11^2$, and that $2(20)(11)\cos D$ is one term. Hence, $2(20)(11)$ is the coefficient of $\cos D$ and cannot be combined with the other numerical terms. Many students will incorrectly try to add the coefficient to both sides. Emphasize that a coefficient multiplies the variable and thus must be divided.

  - Give students who made a mistake time to fix their initial work.
  
  - Students who got it correct can ponder why it is not ambiguous like the Law of Sines was when solving for a missing angle measure. (This is because the second angle possibility for cosine is in Q4 where angles range from 270 degrees to 360 degrees, outside of the realm of possibilities for a triangle angle measure.)
HW problem 4 requires students to remember that the base angles of an isosceles triangle and congruent.

**Day4 – Finding Area Using Trigonometry**

- Students should be comfortable with deriving the formula on p.25 working in groups.
- On Question 5 on p.26, there are two possibilities. Connect this to the ambiguous case in the Law of Sines.
- On p.27, encourage students who are stuck on the second problem to draw a parallelogram and then break it into triangles during the work period. As you circulate, take note of different strategies groups use to answer the second question. Be sure to connect these different strategies during the whole group discussion. During the summary discussion, the whole group should establish that the simplest parallelogram area formula is just double the triangle area formula, which essentially cancels out the ½.

**Trig Applications Review**

- Encourage students to read each problem first and decide which formula will help them solve it. Students should write “area formula,” “Law of Sines,” or “Law of Cosines,” or the exact formula they must use.
Chapter 5: Future Work

This curriculum, as with any curriculum, is imperfect. In future work, more attention should be placed on assessment. Exit tickets should have an additional component where students reflect on their own learning and engage in metacognition. This will greatly benefit the teacher, allowing students to share their perceived misconceptions. This will greatly benefit the student.

In addition, homework assignments should embed spiraled review of past topics. Each homework assignment should include a section of mixed review. This will serve to better prepare students for the unit assessment, where they must decide which strategies to employ to solve a problem.
CURRICULUM STUDY: TRIGONOMETRIC APPLICATIONS

References


GINSBERG ET AL., 2005 FROM THE COMMON CORE HOME PAGE


TRIG APPLICATIONS DAY 1: DERIVING THE LAW OF SINES AND USING IT TO FIND MISSING SIDES

Recall: SOH CAH TOA

What kinds of triangles have a Hyp.?

Right triangles only

Real life rarely has right triangles. We are going to explore non-right triangles using trigonometry.

Sketch triangle ABC and triangle PRQ and label each angle and side.

KNOW THIS:

- The largest side of a triangle is across from its largest ANGLE.
- The smallest side is across from its smallest ANGLE.

**NON-right triangle problems involving a missing side or angle may use either...**

<table>
<thead>
<tr>
<th>Law of Sines</th>
<th>OR</th>
<th>Law of Cosines</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} )</td>
<td></td>
<td>( a^2 = b^2 + c^2 - 2bc \cos A )</td>
</tr>
</tbody>
</table>

Ask students if this reminds them of any formulas they already know.

Today, we will derive the Law of Sines and use it to solve problems. Consider the following problem.

Ask students the meaning of this word.
Ex1 Suppose that two park rangers who are in towers 10 miles apart in a national forest spot a fire that is far away from both of them. Suppose that one ranger recognizes the fire location and knows it is about 4.9 miles from that tower.

With this information and the angles given in the diagram below, the rangers can calculate the distance of the fire from the other tower.

One way to start working on this problem is to divide the obtuse triangle into two right triangles as shown below.

One class of students solved for the length of $\overline{AC}$ using the steps below.

(7PS) To the right of each step, give reasons for why or why not it is correct.

1. $\frac{h}{b} = \sin 29^\circ$  
   Definition of $\sin C = \frac{\text{opp}}{\text{hyp}}$

2. $h = b \sin 29^\circ$  
   Multiply each side by $b$ (some students may reason cross-multiply, which is fine).

3. $\frac{h}{4.9} = \sin 53^\circ$  
   Definition of $\sin B = \frac{\text{opp}}{\text{hyp}}$

4. $h = 4.9 \sin 53^\circ$  
   Multiply each side by 4.9, (or cross multiply)

5. $b \sin 29^\circ = 4.9 \sin 53^\circ$  
   Since step (2) and (4) both equal $h$, the equivalent expressions must also be equal (substitution)

6. $b = \frac{4.9 \sin 53^\circ}{\sin 29^\circ}$  
   Divide both sides by $\sin 29^\circ$
The approach for solving the fire problem (Ex1) on the previous page is almost identical to how the Law of Sines is derived.

Ex2  (In pairs) Explain why each step in the following derivation is correct for \( \Delta ABC \) below

\[
\frac{h}{b} = \sin A \quad \text{Definition of sine of } \angle CAC \]

\[h = b \sin A \quad \text{Cross multiply} \]

\[\frac{h}{a} = \sin B \quad \text{Definition of sine of } \angle BAC \]

\[h = a \sin B \quad \text{Cross multiply} \]

\[b \sin A = a \sin B \quad \text{Substitute quantities equivalent to } h \]

\[\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{Divide both sides by } ab \]

Ex3  (TPS) Modify the above derivation to show that

\[\frac{h}{c} = \sin B \]

\[h = c \sin B \]

\[\frac{h}{b} = \sin C \]

\[h = b \sin C \]

\[\frac{c \sin B}{b} = \frac{b \sin C}{c} \]

The relationships between sides and angles of triangles shown in Ex2 and Ex3 is called the Law of Sines. It can be written in two equivalent forms.

<table>
<thead>
<tr>
<th>The Law of Sines</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} ) \quad \text{and also} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}</td>
</tr>
</tbody>
</table>

Discuss why these forms are equivalent using numerical reasoning.

An important note: when solving problems, you only use _____ of the ratios at a time.
Ex4  (Whole Group) Suppose the park rangers spot a forest fire as indicated in the diagram below. Find the distances from each tower to the fire.

Ex5  (In pairs) In $\triangle ABC$, $m\angle B = 44^\circ$, $m\angle C = 53^\circ$, and $AC = 7$. Find the measure of side $AB$.

$$AB \approx 8.05$$

Ex6  (Individually) Using the same triangle as in Ex5, find the measure of side $BC$.

$$BC \approx 10$$
PRACTICE

Ex7
Find the perimeter of ΔDEF.

Perimeter of ΔDEF ≈ 89.72

Ex8 Two angles of a triangle measure 40° and 56°. If the longest side of the triangle is 28 cm, find the length of the shortest side to the nearest tenth of a centimeter.

18.1 cm
Trig Applications Day 1 TOTD

Two angles of a triangle measure 50° and 75°. If the shortest side of the triangle is 28 cm, find the length of the longest side to the nearest tenth of a centimeter.

35.3
Trig Applications Day 1 HOMEWORK

1) Triangle SEA is shown below. Find the measure of \( x \), the distance from A to S.

\[ x \approx 30.0 \text{ miles} \]

2) In a triangle, two angles measure \( 14^\circ \) and \( 110^\circ \). The side opposite the \( 14^\circ \) angle measures 9.4 units. Find the length of the longest side.

\[ 36.5 \]

3) Triangle ABC is shown below.

a) If you were tasked to find the length of \( BC \) using the Law of Sines, what is the minimal amount of information you would need to be given?

You must be given 2 angles and 1 side (6 or c).

b) Using your answer to part a), explain how you would use the given information to find the length of side \( BC \).

Find the remaining angle by subtracting from 180°.

Set up the Law of Sines ensuring that one of the ratios includes the given side and the other ratio includes \( a \).

Finally, solve for \( a \).
DAY 2: LAW OF SINES & MISSING ANGLES - THE AMBIGUOUS CASE

I. Exploring the SSA ambiguous case

Consider the following example...

In $\triangle ABC$, $a = 10$, $b = 16$ and $m\angle A = 150^\circ$. Find $\sin B$.

$$\frac{\sin 150^\circ}{10} = \frac{\sin B}{16}$$

$$\sin B = \frac{16 \sin 150^\circ}{10} = 0.8$$

What if the question above asked you to find $m\angle B$ to the nearest degree?

Angle whose sine is 0.8

$$\sin^{-1}(0.8) \approx 53.13^\circ$$

Is this angle measure possible? Explain.

No! $53.13^\circ + 150^\circ = 203.13^\circ$

All triangles must have angles that sum to 180°

The Law of Sines is AMBIGUOUS (may have different answers) when solving for angles. This occurs when given two sides and the non-included angle of a triangle (SSA). There are either 0, 1, or 2 possible triangles.

To show this, we will fix non-included angle $A$, and sides $b$ and $a$.

(TIPS): If we fix these three parts, which parts are you allowed to change?

The altitude $h$ is also labeled. Use a compass to explore each case.

Case 1: The third side is smaller than the altitude ($a < h$)

Conclusion:

When $a$ is smaller than $h$, there are 0 possible triangles
Case 2: The third side is in between the lengths h and b \((h < a < b)\)

Case 3: the third side is greater than b \((a > b)\)

II. Practice Problems

Exercise #1: In triangle \(ABC\), which is shown below but not drawn to scale, it is known that \(AC = 6\), \(AB = 10\), and \(m \angle B = 36^\circ\). Determine all possible values for \(m \angle C\) to the nearest tenth.

\[
\frac{\sin C}{10} = \frac{\sin 36^\circ}{6}
\]

\[
\sin C = \frac{10 \sin 36^\circ}{6}
\]

\[
C = \sin^{-1} \left( \frac{10 \sin 36^\circ}{6} \right) = 78.4^\circ
\]

or

\[
C = 180 - 78.4^\circ = 101.6^\circ
\]

Check:

\[
101.6 + 36 = 137.6
\]

Needs to be < 180°
Exercise #2: In triangle $ABC$, which is shown below but not drawn to scale, it is known that $AC = 10$, $AB = 14$, and $m\angle C = 80^\circ$.

(a) Solve an equation to find all possible values for the measure of $B$ to the nearest tenth.

$$\frac{\sin B}{10} = \frac{\sin 80^\circ}{14}$$

$$\sin B = \frac{10 \sin 80^\circ}{14} \Rightarrow m\angle B = 44.7^\circ \text{ or } m\angle B = 135.3^\circ$$

Reject

(b) Considering the measures the angles of any triangle must sum to be $180^\circ$, why must we reject the obtuse solution from part (a)?

Since angles of triangles must sum to $180^\circ$, the obtuse angle $135.3^\circ$ plus the given angle $80^\circ$ sum to $215.3^\circ$, which is larger than $180^\circ$ and must be rejected.

Exercise #3: Explain why the triangle shown below cannot exist by finding all possible values for $m\angle C$.

$$\frac{\sin C}{18} = \frac{\sin 52^\circ}{12}$$

$$\sin C = \frac{18 \sin 52^\circ}{12} \approx 1.18$$

$$C = \sin^{-1}(1.18) = \text{ERROR}!!$$

Why??

Remember: $\sin C = \frac{\text{opp}}{\text{hyp}} \leq \text{bigger #} |

\sin \text{e (and cosine) can not be larger than 1}!$

Exercise #4: In $\triangle DEF$, $m\angle E = 72^\circ$, $DE = 12$, and $DF = 15$. How many triangles are possible given this information?

(1) 1

(2) 2

(3) 3

(4) 0

Talk about why you can automatically eliminate this
Trig Applications Day 2 TOTD

Given \( \triangle ABC \) such that \( A = 30^\circ, \ a = 20, \) and \( b = 16, \) determine the number of possible triangles.

\[
\sin B = \frac{16 \sin 30}{20}
\]

\[ B \approx 24^\circ \quad \text{or} \quad B \neq 156^\circ \rightarrow \text{Reject} \]

\[ 150 + 30 = 180^\circ \]

Not possible!

1 possible triangle

---

Trig Applications Day 2 TOTD

Given \( \triangle ABC \) such that \( A = 30^\circ, \ a = 20, \) and \( b = 16, \) determine the number of possible triangles.
Trig Applications Day 2 HOMEWORK

SKILLS

Diagrams given in problems are not drawn to scale and angles that appear acute may in fact be obtuse and vice versa.

1. In \( \triangle CDE \) shown below \( CE = 10, CD = 8, \) and \( m \angle E = 40^\circ \). How many possible values exist for \( m \angle D \)?

   (1) 1
   (2) 2
   (3) 3
   (4) 0

2. In \( \triangle ABC \) shown \( m \angle C = 25^\circ, c = 15, \) and \( b = 20 \). Angle \( B \) is

   (1) acute only
   (2) right only
   (3) obtuse only
   (4) acute or obtuse

3. For which value of \( f \) shown below will there be no solution for angle \( E \) in triangle \( DEF \)?

   (1) 9
   (2) 10
   (3) 11
   (4) 5

4. Accurate to the nearest tenth the largest possible value of \( m \angle N \) in the diagram below is

   (1) 49.8
   (2) 130.2
   (3) 37.3
   (4) 105.6

5. How many triangles can be formed in which the shortest side measures 9 inches, the longest side measures 14 inches and the smallest angle measures 48°?

   (1) 1
   (2) 2
   (3) 3
   (4) 0
DAY 3: THE LAW OF COSINES

In the past two lessons, we used the Law of Sines to determine a missing side or angle. In each situation, we involved two sides and two angles where one side or angle was missing. For problems with different combinations of angle measures and side lengths being known, a second property is helpful: The Law of Cosines.

I. Getting comfortable with a new formula

The Law of Cosines states that in any triangle $ABC$ with sides of length $a$, $b$, and $c$,

$$c^2 = a^2 + b^2 - 2ab \cos C$$

1. If you are solving for a missing side, what number of side(s) and angle(s) measurements must be known?

   Since the formula involves 3 different sides and 1 angle, if you are finding a missing side, 2 sides and 1 angle must be known.

2. If you are solving for a missing angle, what number of side(s) and angle(s) measurements must be known?

   3 sides and 0 angles

When we are solving for an angle or a side, always rearrange the formula to include the missing side or angle's letter first. This means we could use the above formula to solve for either side $c$ or angle $C$.

3. Rearrange the letters in the formula to set up to solve for side $a$ or angle $A$.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

4. Rearrange the letters in the formula to set up to solve for the side $b$ or angle $B$.

$$b^2 = a^2 + c^2 - 2ac \cos B$$

5. If you were given triangle $XYZ$, set up the formula to solve for missing side $y$ or angle $Y$.

$$y^2 = x^2 + z^2 - 2xz \cos Y$$
II. Deriving the Law of Cosines

Derivations of the Law of Cosines depend on whether the angle under consideration is acute or obtuse. In our first setup, $\triangle ABC$ contains obtuse $\angle B$ and $h$ is the altitude of the triangle drawn from vertex $C$.

1. Explain why each step in the following derivation is true. The reason for (5) is given.

   (1) In right triangle $BCE$,
   
   \[
   x^2 + h^2 = a^2 \quad \text{Rythagorean Theorem} \\
   h^2 = a^2 - x^2 \quad \text{Subtract $x^2$ from both sides}
   \]

   (2) In right triangle $ACE$,
   
   \[
   h^2 + (x+c)^2 = b^2 \quad \text{Rythagorean Theorem} \\
   h^2 = b^2 - (x+c)^2 \quad \text{Subtract $(x+c)^2$ from both sides}
   \]

   (3) $a^2 - x^2 = b^2 - (x+c)^2$ Each quantity is equivalent to $h^2$
   
   \[
   a^2 - x^2 = b^2 - (x^2 + 2xc + c^2) \\
   a^2 - x^2 = b^2 - x^2 - 2xc - c^2 \quad \text{solve for } b^2
   \]

   (4) $b^2 = a^2 + c^2 + 2cx$
   
   \[
   \text{Next we find } x
   \]

   (5) In right triangle $BCE$,
   
   \[
   \cos(m\angle CBE) = \frac{x}{a} \quad \text{Definition of cosine} \\
   x = a \cos(m\angle CBE) = -a \cos B \quad \text{Reason: Since angle B is obtuse but less than } 180^\circ, \text{ the cosine of angle B is negative.}
   \]

   (6) $b^2 = a^2 + c^2 - 2ac \cos B$ 
   
   \[
   \text{Substitute } -a \cos B \text{ in place of } x \text{ in step (4)}
   \]
For the case with an acute angle, you will be guided through the proof. Complete this in groups of 3. Discuss your work as you go.

Begin with $\triangle ABC$ with altitude $k$ drawn from vertex $C$.

1. The altitude separates $\triangle ABC$ into two right triangles. Use the Pythagorean theorem to write two equations, one relating $k$, $b$, and $c-x$, and another relating $a$, $k$, and $x$.

\[
(1) \quad k^2 + (c-x)^2 = b^2
\]
\[
(2) \quad k^2 + x^2 = a^2
\]

2. Notice that both equations contain $k^2$. Solve each equation for $k^2$.

\[
(1) \quad k^2 = b^2 - (c-x)^2
\]
\[
(2) \quad k^2 = a^2 - x^2
\]

3. Since both of the equations in Question 2 are equal to $k^2$, they can be set equal to each other. Set the equations equal to each other to form a new equation.

\[
b^2 - (c-x)^2 = a^2 - x^2
\]

4. Notice that the equation in Question 3 involves $x$. However, $x$ is not a side of $\triangle ABC$. We will try to rewrite the equation in Question 3 so that it does not include $x$. Begin by expanding $(c-x)^2$.

\[
b^2 - (c^2 - 2cx + x^2) = a^2 - x^2
\]
\[
b^2 - c^2 + 2cx - x^2 = a^2 - x^2
\]
5. Solve the equation in Question 4 for \(b^2\).

\[
b^2 = \frac{a^2 - x^2}{c^2 - x^2 + c^2 - 2cx + x^2}
\]

\[
b^2 = a^2 + c^2 - 2cx
\]

6. The equation in Question 5 still involves \(x\). To eliminate \(x\) from the equation, we will attempt to substitute an equivalent expression for \(x\). Write an equation involving both \(\cos B\) and \(x\).

\[
\cos B = \frac{x}{a}
\]

7. Solve the equation from Question 6 for \(x\).

\[
x = a\cos B
\]

8. Substitute the equivalent expression for \(x\) into the equation from Question 5. The resulting equation contains only sides and angles from \(\triangle ABC\). This equation is the Law of Cosines.

\[
b^2 = a^2 + c^2 - 2ac\cos B
\]
III. Using the Law of Cosines to Solve Problems

Exercise #1: If \( m \angle C = 90^\circ \) then the first form of the Law of Cosines results in what famous theorem? Justify your answer by substituting \( 90^\circ \) for \( C \).

\[
\begin{align*}
C^2 &= a^2 + b^2 - 2ab \cos 90^\circ \\
C^2 &= a^2 + b^2 - 2ab \cos 0^\circ \\
\end{align*}
\]

\( \cos 90^\circ = 0 \)

Exercise #2: In \( \triangle ABC \), below, \( AB = 8, \ BC = 13, \) and \( m \angle B = 110^\circ \). Find the length of \( AC \) to the nearest tenth.

\[
\begin{align*}
b^2 &= a^2 + c^2 - 2ac \cos B \\
b^2 &= 13^2 + 8^2 - 2(13)(8) \cos 110^\circ \\
\sqrt{b^2} &= \sqrt{304.1401 \times 0.898} \\
b &= 17.4
\end{align*}
\]

Exercise #3: In triangle \( \triangle DEF \), below, \( DE = 11, \ EF = 15, \) and \( DF = 20 \). Determine \( m \angle D \) to the nearest degree.

\[
\begin{align*}
ed^2 &= e^2 + f^2 - 2ef \cos D \\
15^2 &= 20^2 + 11^2 - 2(20)(11) \cos D \\
225 &= 521 - 440 \cos D \\
\cos D &= \frac{1672}{440} \\
D &= \cos^{-1} \left( \frac{7}{10} \right) = 48^\circ
\end{align*}
\]

Exercise #4: A triangle has three sides that measure 8, 14, and 20 centimeters.

Find the measure of the largest angle of this triangle to the nearest tenth of a degree.

\[
\begin{align*}
20^2 &= 8^2 + 14^2 - 2(8)(14) \cos C \\
140 &= -224 \cos C \\
\therefore \quad m \angle C &= 128.7^\circ
\end{align*}
\]

\( \times \) students may use different angle labels.
Trig Applications Day 3 TOTD

1) Given \( \triangle XYZ \), make a sketch and label all angles and sides. Then, write the correct setup to solve for angle \( Z \) using the Law of Cosines

\[
\begin{align*}
Z^2 &= X^2 + Y^2 - 2XY \cos Z
\end{align*}
\]

2) If angle \( Y \), angle \( X \), and sides \( x \) and \( y \) are known, explain whether or not the Law of Cosines could be used to solve for angle \( Z \).

The Law of Cosines requires all 3 side lengths to determine a missing angle, and only 2 are given.

(The measure of \( \angle Z \) could easily be found by subtracting \( 180^\circ - m\angle Y - m\angle X \) however.)

3) Find \( BC \).

\[
\begin{align*}
C^2 &= 21^2 + 30^2 - 2(21)(30)\cos 123^\circ
\end{align*}
\]

\[ C \approx 415 \]
Trig Applications Day 3 HOMEWORK

SKILLS

1. In each of the triangles below, a S.A.S. scenario is given. Find the value of $x$ in each case to the nearest tenth using the Law of Cosines.

(a)

![Triangle A](image1)

$x = 11.9$

(b)

![Triangle B](image2)

$x = 15.6$

2. In each of the triangles below, a S.S.S. scenario is given. Using the Law of Cosines, find the measure of $\theta$ as marked in each triangle accurate to the nearest degree.

(a)

![Triangle C](image3)

$\theta = 35^\circ$

(b)

![Triangle D](image4)

$\theta = 127^\circ$

3. In a triangle whose side lengths measure 20, 21, and 29 inches respectively, the largest angle is

(1) acute  (3) obtuse

(2) right  (4) either acute or obtuse

2
DAY 4: FINDING AREA USING TRIGONOMETRY

In the first three lessons, we developed techniques and formulas for finding missing sides or angles of triangles. It is often useful to know the exact amount of space a figure contains. We will develop a more sophisticated area formula for determining the area of a triangle. Let's begin with the formula you are already familiar with.

Recall:  \[
\text{Area of a Triangle} = \frac{1}{2} \text{ base} \times \text{ height}
\]

\[
= \frac{1}{2} (6)(8)
= 24 \text{ square units}
\]

In the above example, we used the fact that a right triangle's height is always known. What if we want to find the area of a triangle when the height is not known? With a partner, follow each step to arrive at the new formula.

1. Draw altitude from vertex A to side BC and label its length \(h\). Notice that \(h\) is the height of \(\triangle ABC\).

2. Use your familiar triangle area formula to write the area of this triangle using \(a\) and \(h\).

\[
\text{Area} = \frac{1}{2} \cdot a \cdot h
\]

3. Since \(h\) is not the length of a side of \(\triangle ABC\), we must work to represent \(h\) using relationships between given sides and angles. To do this, write an equation for \(\sin C\) relating \(h\) and side \(b\).

\[
\sin C = \frac{h}{b}
\]

4. Solve this equation for \(h\).

\[
h = b \sin C
\]

5. Substitute the expression from part 4 into your area formula in part 2. Verify that you have the same formula as another group and this this matches the formula on the next page.

\[
\text{Area} = \frac{1}{2} ab \sin C
\]
II. Getting comfortable using our new formula.

\[ \text{Area} = \frac{1}{2} ab \sin C \]

The formula you derived is able to find the area of a triangle if sides \(a\), and \(b\), and angle \(C\) are known.

1. What is important to note about how angle \(C\) is positioned in relation to sides \(a\) and \(b\)?

   \(\text{Angle } C \text{ is formed by sides } a \text{ and } b\)
   \(\text{(It is the included angle)}\)

2. If given side measures \(a\) and \(c\), what angle would need to be given in order to apply the area formula? Modify the above area formula to show this.

   \[ \text{Area} = \frac{1}{2} ac \sin B \]

3. If given angle \(A\), what two sides would also need to be given in order to apply the area formula? Modify the above area formula to show this.

   \[ \text{We would need sides } b \text{ and } c \]
   \[ \text{Area} = \frac{1}{2} bc \sin A \]

4. Modify the area formula to represent the area of a different triangle, \(\triangle ACE\). Compare your response to another group's. What did you notice? Together, determine at least one additional area formula for \(\triangle ACE\).

   \[ \text{Area} = \frac{1}{2} ac \sin C = \frac{1}{2} ac \sin E = \frac{1}{2} ce \sin A \]

5. Is it possible to use the area formula to solve for a missing side or angle of a triangle? Explain your thinking.

   \(\text{It is possible. If you are given two sides and the area is possible to solve for the missing angle they form. If given the area, one side and one angle, you can solve for the missing side that forms the angle.}\)
III. Using the area formula to solve problems

Ex: Find the area of \( \triangle DEF \) if \( DE = 14 \), \( EF = 9 \), and \( \angle E = 30^\circ \).

\[
\text{Area} = \frac{1}{2} \cdot 14 \cdot 9 \cdot \sin 30^\circ = 31.5
\]

Ex: The adjacent sides of parallelogram ABCD measure 12 and 15. The measure of the angle of the parallelogram is 135°. Find the area of the parallelogram to the nearest tenth.

\[
\text{Area} = 127.3
\]

Ex: Mr. Wilson is trying to determine the amount of paint he will need for the triangular portion in front of his house. A gallon of paint he will use will cover 150 square feet. If the portion he must cover has the shape of an isosceles triangle, shown below, with legs 36 feet and base angles of 30°, determine the minimum number of gallons of paint Mr. Wilson will need. Note that Mr. Wilson can only buy full gallons of paint.

\[
\text{Area} \approx 561.1844617
\]

\[
\frac{561.184467}{150} = 3.7...
\]

Mr. Wilson needs 4 gallons of paint.
Trig Applications Day 4 TOTD

1) In general, what information needs to be given in order to determine the area of a triangle using the formula \( \text{Area} = \frac{1}{2}ab\sin C \)?

2) Determine the area of the following triangle.

\[\text{Area} = 107.7\]

3) Create your own real world problem involving finding the area of a triangle using the formula from today’s lesson and solve it.

Responses will vary
Trig Applications Day 4 HOMEWORK

1. Modify the diagram, wording, and work you did in the first part of today’s lesson to derive the area formula \( \text{Area} = \frac{1}{2} bc \sin A \).

\[
\begin{align*}
\text{Area} &= \frac{1}{2} c \cdot h \\
\sin A &= \frac{h}{b} \\
h &= b \cdot \sin A \\
\text{Area} &= \frac{1}{2} c \cdot b \sin A \\
\text{Area} &= \frac{1}{2} b c \sin A
\end{align*}
\]

2. Find the area of the triangle with the given measurements to the nearest tenth.
   a. In \( \triangle ABC \), \( b = 14.6 \), \( c = 12.8 \), \( m \angle A = 56^\circ \)

\[
\text{Area} = 77.5
\]

b. In \( \triangle PQR \), \( p = 212 \), \( q = 287 \), \( m \angle R = 124^\circ \)

\[
\text{Area} = 25221.0
\]

c. In \( \triangle ABC \), \( b = 7 \), \( c = 8 \), \( \sin A = \frac{3}{5} \)

\[
\text{Area} = 33.6
\]
3. Find the area of a parallelogram if the measures of two adjacent sides are 40 feet and 24 feet and the measures of one angle of the parallelogram is 30 degrees.

4. The following diagram shows the peak of a roof that is in the shape of an isosceles triangle. A base angle of the triangle measures 50° and each side of the roof is 20.4 feet. Determine to the nearest tenth of a square foot the area of the triangular region.
1. The area of an equilateral triangle, in simplest radical form, whose side lengths measure 10 is

(1) $50\sqrt{2}$  (3) $25\sqrt{3}$
(2) $22\sqrt{7}$  (4) $100\sqrt{2}$

2. Vivian would like to create a window in the shape of an equilateral triangle whose area is 100 square inches. To the nearest tenth of an inch, how long should each side of this triangle be?

(1) 17.6  (3) 15.2
(2) 9.7  (4) 22.8

$$\frac{100}{\sin 60^\circ} = \frac{a^2}{2}$$

3. In $\triangle ABC$, $a = 4$, $b = 2$ and $\sin A = \frac{6}{7}$. Which of the following must be the value of $\sin B$?

(1) $\frac{3}{7}$  (3) $\frac{2}{3}$
(2) $\frac{1}{2}$  (4) $\frac{12}{7}$

4. Rohan takes measurements of triangle $ABC$ and finds that $AC = 5$, $AB = 8$ and $\angle C = 46^\circ$. How many triangles exist that could have the measurements that Rohan took?

(1) 1  (3) 3
(2) 2  (4) 0

$$\frac{\sin B}{5} = \frac{\sin 46^\circ}{8}$$

$m\angle B = 26.7^\circ$ or $m\angle B / 199.2^\circ$

$199.2^\circ + 46^\circ = 245^\circ$

Impossible!
Given the diagram below, then the value of \( x \), to the nearest tenth, is

\[
\begin{align*}
(1) \ & 32.8 \\
(2) \ & 13.4 \\
(3) \ & 30.1 \\
(4) \ & 18.7
\end{align*}
\]

\[ \text{Law of Cosines} \]

---

Marissa would like to create a flower patch in the shape of an isosceles triangle. She has enough room so that the triangle has dimensions as shown below. Determine the area of Marissa’s triangle to the nearest square foot.

\[ \text{Area formula} \]

\[ 68 \text{ square feet} \]

---

Explain why the triangle shown below cannot exist.

Responses will vary.

Students should establish that the sine of the angle opposite the side of length 18 is greater than 1.
A triangle has side lengths of 8, 13, and 18 inches. Determine the smallest angle of this triangle to the nearest degree.

\[ \text{Law of Cosines} \]

\[ 8^2 = 13^2 + 18^2 - 2(13)(18)\cos x \]

\[ x = 24^\circ \]

In the diagram shown below, points D, E, and F are collinear. If \( m\angle D = 22^\circ \), \( m\angle CEF = 52^\circ \) and \( DE = 25 \) then find the length of \( CF \) to the nearest tenth.

1. Determine all missing \( \angle \) measures.
2. Can't find \( CF \) directly. Find the length of reflexive side CE using Law of Sines

\[
\sin 22^\circ = \frac{\sin 38^\circ}{y} \\
y = 18.73032967
\]

3. Use the Law of Sines to find \( CF \) OR set up a trig ratio such as \( \sin 32^\circ = \frac{x}{18.73032967} \)

\[ x = 14.8 \]

Jonathan has created a triangular room in his house that has side lengths of 12 feet, 16 feet and 20 feet. He would like to cover it with flooring that costs $3.25 per square foot. Assuming there is no wasted flooring, calculate, to the nearest dollar, how much Jonathan will need to spend on flooring.

- Need an angle so that you can use the area formula.
- Use Law of Cosines to find a missing angle.

\[ \text{Area} = 96 \]

\[ 96 \times 3.25 = 312 \]
Trig Applications Unit Test

Part I – Multiple Choice

Each question is worth 2 points. No partial credit will be awarded.

1. In \( \triangle ABC \), \( AB = 12 \), \( BC = 16 \) and \( m \angle B = 72^\circ \). Which of the following represents the area of \( ABC \) to the nearest integer?

   (1) 91  
   (2) 96  
   (3) 78  
   (4) 183

   Answer: 91

2. An isosceles triangle has legs with lengths of 24 inches and base angles that measure 56\(^\circ\). To the nearest square inch, which of the following is the area of this isosceles triangle?

   (1) 184  
   (2) 267  
   (3) 239  
   (4) 108

   Answer: 267

3. A triangle whose area is 15 has two sides of length 6 and 10. If the angle created by these two sides is obtuse, then its measure must be \( \frac{1}{2} \cdot 6 \cdot 10 \cdot \sin A \).

   (1) 120\(^\circ\)  
   (2) 135\(^\circ\)  
   (3) 30\(^\circ\)  
   (4) 150\(^\circ\)

   Answer: 135\(^\circ\)
4. In \( \triangle DEF \), \( EF = 18 \), \( m \angle D = 75^\circ \) and \( m \angle F = 20^\circ \). To the nearest tenth, the length of \( DE \) is

(1) 4.8  
(2) 5.0  
(3) 6.4  
(4) 7.8

5. In \( \triangle MNP \), \( m \angle N = 30^\circ \), \( MN = 10 \) and \( MP = 6 \). Given this information, angle \( P \) could be

(1) acute only  
(2) right only  
(3) acute or obtuse  
(4) obtuse only

6. In the triangle shown below, what is the measure of the smallest angle, to the nearest degree?

(1) 29\(^\circ\)  
(2) 47\(^\circ\)  
(3) 76\(^\circ\)  
(4) 22\(^\circ\)
7 In \( \Delta ABC \) shown below, it is known that angle \( B \) is obtuse. Find the measure of angle \( B \) to the nearest degree.

\[
\frac{\sin B}{14} = \frac{\sin 36^\circ}{10}
\]

\[
\sin B = \frac{14 \sin 36^\circ}{10}
\]

\[
\measuredangle B = \sin^{-1}\left( \frac{14 \sin 36^\circ}{10} \right) = 55^\circ
\]

**OR** \( \measuredangle B = 125^\circ \)

8 A radar is tracking the location of a plane. Initially the plane is at a distance of 6.5 miles from the radar. After the radar rotates through an angle of 115\(^\circ\), the plane is now at a distance of 3.2 miles from the radar. What straight line distance, to the nearest tenth of a mile, has the plane flown between these two readings?

\[
x^2 = 3.2^2 + 6.5^2 - 2(3.2)(6.5)\cos 115^\circ
\]

\[
x = 8.5 \text{ miles}
\]
Part III – Extended Response

Each question is worth 4 points. Partial credit may be awarded.

9 In \( \triangle ABC \) shown below, not drawn to scale, it is given that \( m\angle C = 44^\circ \), \( AC = 10 \) and \( AB = 8 \). Find all possible areas of \( ABC \) to the nearest integer.

1. Find \( m\angle B \)
   \[
   \frac{\sin B}{10} = \frac{\sin 44^\circ}{8} 
   \]
   \( m\angle B = 60.26^\circ \) or \( m\angle B = 119.74^\circ \)

2. Possible \( m\angle C \)
   \( m\angle C = 75.74^\circ \) or \( m\angle C = 16.26^\circ \)

3. Possible Areas
   \[
   \text{Area} = \frac{1}{2} (8)(10) \sin(75.74^\circ) \quad \text{or} \quad \text{Area} = \frac{1}{2} (8)(10) \sin(16.26^\circ)
   \]
   \[= 39 \]

1 In quadrilateral \( ABCD \) it is known that \( AB = 16 \), \( BC = 24 \), \( CD = 36 \), \( m\angle C = 32^\circ \) and \( m\angle ABD = 64^\circ \).

Determine, to the nearest tenth the length of \( AD \).

\[
y^2 = 24^2 + 36^2 - 2(24)(36) \cos 32^\circ
\]
\[y \approx 20.164\]

\[
x^2 = 16^2 + 20.164^2 - 2(16)(20.164) \cos 64^\circ
\]
\[x = 19.5\]