Lesson 1: Connection to Linearity

A. George works at an ice cream stand during the summer. He gets paid an initial amount each day and then an hourly rate. The table below represents his daily pay, \( G \), after \( h \) hours of work.

<table>
<thead>
<tr>
<th>Hours of work</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daily Pay (in dollars)</td>
<td>15</td>
<td>23</td>
<td>31</td>
<td>39</td>
<td>47</td>
<td>55</td>
<td>63</td>
<td>71</td>
<td>79</td>
</tr>
</tbody>
</table>

1. Complete the table above to continue the pattern of \( (h, G) \) coordinates.

2. Describe the relationship between the number of hours George works, \( h \), and the amount of money he earns each day, \( G \). Include the values for the initial pay and hourly pay in your response.

   George earns $15 each day initially. He also earns an additional $8 for each hour he works.

3. a. Write an equation to represent George’s daily pay, \( G \), after \( h \) hours of work.

   \[ G = 15 + 8h \]
   
   or
   
   \[ G = 8h + 15 \]

b. Explain how your equation in (a) represents the relationship you described in question 2.

   \[ G = 15 + 8h \]

   money earned  \( \downarrow \) initial pay
   
   hourly pay  \( \uparrow \) number of hours worked
   
   money earned  \( \downarrow \) initial pay

   c. Determine the amount of money George would earn if he worked a 12 hour shift.

   \[ G = 15 + 8(12) \]
   
   \[ G = 111 \]
   
   $111
4.

a. Graph George’s daily income relationship on the grid below.

\[
G(10) = 15 + 8(10) = 95
\]

b. Explain how your graph in represents the relationship you described in question 2.

\[
\text{y-int at } (0, 15) \Rightarrow \text{ initial daily pay of } $15.
\]

\[
\text{Slope of } 8 \Rightarrow \text{ hourly pay of } $8
\]

c. Did you connect your data points on your graph? In this context, would it make sense to do so?

Yes. George can work a fraction of an hour (like 2.5 hrs). So it makes sense to connect the data points to include all fractional hourly values.

d. Describe what the coordinate point (1.5, 27) means in this situation.

For 1.5 hrs of work, George earns $27.
ARITHMETIC SEQUENCES USING SITUATED TASKS

B. Michelle works at a lemonade stand by the beach. She is paid an initial amount each day and then earns commission from each drink that she sells. The table below represents her daily pay, $M$, after $n$ lemonades are sold.

<table>
<thead>
<tr>
<th>Lemonades Sold</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daily Pay (in dollars)</td>
<td>15.5</td>
<td>18</td>
<td>20.5</td>
<td>23</td>
<td>25.5</td>
<td>28</td>
<td>30.5</td>
<td>33</td>
</tr>
</tbody>
</table>

1. Complete the table above to continue the pattern of $(n, M)$ coordinates.

2. Describe the relationship between the number of lemonades Michelle sells, $n$, and the amount of money she earns each day, $M$. Include the values for the initial pay and amount earned per lemonade sold in your response.

   Michelle initially earns $13 each day. She also earns $2.50 for each lemonade she sells.

3. 
   a. Write an equation to represent Michelle's daily pay, $M$, after $n$ lemonades are sold.

      $M = 13 + 2.5n$ 
      or 
      $M = 2.5n + 13$

   b. Explain how your equation in (a) represents the relationship you described in question 2.

   c. Compare Michelle's daily earnings to George's. What similarities exist and how are the two situations different?

   Both have an initial pay but George's is $15 while Michelle's is $13. They both have additional pay but George's is hourly and Michelle's is based on how much lemonade she sells.
4.

a. Graph Michelle’s daily income relationship on the grid below.

b. Explain how your graph in represents the relationship you described in question 2.

\[ 13 + 2.5(10) = 38 \]

Pay Earned (in dollars)

\[ (0, 13) \] (1, 15.5)

\[ \Delta y = 2.5 \]

\[ \Delta x = 1 \]

\[ y \text{-int at } (0, 13) \rightarrow \text{initial daily pay of } \$13 \]

\[ \text{slope of } 2.5 \rightarrow \text{commission of } \$2.50 \text{ per lemonade sold} \]

c. Did you connect your data points on your graph? In this context, would it make sense to do so?

No. Michelle cannot sell a fraction of a lemonade.

d. Describe what the coordinate point (1.5, 16.75) would mean in this situation. Is this realistic?

One and a half lemonades sold would earn Michelle $16.75. This is not realistic because you cannot sell half of a lemonade.
C. Both George’s and Michelle’s pay are examples of linear relationships. Michelle’s is also an example of an arithmetic sequence. Based on your work in parts A and B, what do you think makes a linear situation an arithmetic sequence? Why is George’s pay not an example of an arithmetic sequence?

An arithmetic sequence has a domain of positive integers (sometimes including 0) only. Linear situations can have domains of all real numbers. George can work for a non-integer value number of hours, so his relationship is not an arithmetic sequence.

Extensions/Homework

1. Jane works at a nail salon and is paid $12 an hour. Would her pay represent an arithmetic sequence? Explain.

No. The domain of her hours worked could include non-integers (like 4.25 hrs).

2. Write an equation for the relationship in problem 1. Compare this relationship to George’s pay relationship in part A.

\[ J = 12h \]

Jane earns more than George per hour but does not make any initial amount like he does.

3. Would the relationship shown in the table below represent an arithmetic sequence?

<table>
<thead>
<tr>
<th>Hours worked</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pay ($)</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
</tr>
</tbody>
</table>

No. This relationship is not linear. It is exponential.
Lesson 2: Explicit Formulas

A. Michael, Michelle's brother, works at an iced tea stand near his sister's stand by the beach. Michael’s boss pays him more each day but he earns less for each drink he sells. He earns a daily pay of $20 and then earns $1 for each iced tea he sells.

1. Complete the table below to represent how much Michael earns in one day of sales.

<table>
<thead>
<tr>
<th>Iced Teas Sold</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daily Pay (in dollars)</td>
<td>20</td>
<td>21</td>
<td>22</td>
<td>23</td>
<td>24</td>
<td>25</td>
<td>26</td>
<td>27</td>
</tr>
</tbody>
</table>

2. 
   a. Write an equation to represent the amount of money Michael earns, $M$, in one day after he sells $t$ iced teas.

   \[ M = 20 + 1t \]

   b. Label how each variable and number in your equation represents this situation.

   \[ M = \text{initial daily pay} + \text{number of iced teas} \]

3. Does Michael's daily pay relationship represent an arithmetic sequence or is it just a linear relationship? Explain.

   Arithmetic sequence. The domain is possible integers since you can't buy part of an iced tea.
4.  

a. Graph Michael's daily income relationship on the grid below.

b. Explain how your graph represents your equation from part 2a.

\[ y - \text{int at } (0, 20) \Rightarrow \text{initial pay } \text{if } \$20 = b \]

\[ (0, 20) \text{ and } (1, 21) \]  \[ \frac{\Delta y}{\Delta x} = 1 \]  \[ \text{Slope } = 1 \Rightarrow \$1 \text{ per iced tea } = m \]

c. Did you connect your data points on your graph? In this context, would it make sense to do so?

No. Michael cannot sell a fraction of an iced tea.
B. Any arithmetic sequence can be written as an equation in the form

\[ a_n = a_1 + (n - 1)d \]

This is often how reference sheets display the generic equation. The variables are defined as follows:

- \( a_n \) is the \( n^{th} \) term of the sequence. For example, \( a_6 \) is the 6\(^{th} \) term.
- \( n \) represents the number of the current term. For \( a_6, n = 6 \).
- \( a_1 \) is the term when \( n = 1 \) which is usually the first term of the sequence.
- \( d \) is the common difference. It represents how much the sequence changes from one term to the next.

1. Suppose we apply this new equation format to Michael’s daily pay relationship from part A. What would \( a_6 \) represent in this case?

\( a_6 \) would be the pay Michael earns in a day if he sells 6 iced teas.

2. Michael says that \( a_1 \) represents the y-intercept. Whereas Michelle says that \( a_0 \) represents the y-intercept. Who do you agree with? Explain.

\( a_0 \) \( \Rightarrow \) the y-intercept is when \( x=0 \) not 1.

3. What is the value of \( d \) for Michael’s pay situation? What is another name we typically use for this value?

\( d = 1 \) \( \Rightarrow \) slope

4. Write an equation using the new format to represent Michael’s daily pay.

\[ a_n = 21 + (n-1)/1 \]

5. Simplify your equation from part 4. How does this equation relate to the one you wrote in part 2a?

\( a_n = 21 + n - 1 \)

\( a_n = 20 + n \) \( \Rightarrow \) equivalent to \( M = 20 + 1t \), just with different variables
C. Arithmetic sequences can be written as equations using the traditional linear format of \( y = mx + b \) or the new format \( a_n = a_1 + (n - 1)d \).

1. Which variable in the linear equation corresponds to the \( a_n \) variable from the new format?

2. Which variable in the new format corresponds to the \( m \) variable from the linear format?

3. Is there a variable in the new format that corresponds to the variable \( b \) from the linear format? If not, how could we write it using the new notation?

No. Should be \( a_0 \)

D. For each sequence shown below, write an equation using the traditional \( y = mx + b \) format and an equation using the new \( a_n = a_1 + (n - 1)d \). Simplify all equations completely and draw arrows to show the relationship between the two equations.

1. \( y \)-intercept of 6, slope of \(-3\).

2. \( a_1 = \frac{1}{2}, d = 2\frac{1}{2} \)

3. 

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>14</td>
<td>9</td>
<td>4</td>
<td>(-1)</td>
<td>(-6)</td>
</tr>
</tbody>
</table>

E. For each sequence from part D, determine the value of \( a_{10} \).

1. \( a_{10} = -3(10) + 6 \)
   \[ a_{10} = -30 + 6 = -24 \]

2. \( a_{10} = 2\frac{1}{2}(10) - 2 \)
   \[ a_{10} = 25 - 2 = 23 \]

3. \( a_{10} = -5(10) + 19 \)
   \[ a_{10} = -50 + 19 = -31 \]
Extensions/Homework

1. Use the equation of an arithmetic sequence show below to answer the following questions.

\[ a_n = 6 + (n - 1)(4) \]

a. What is the first term, \( a_1 \)? What is the common difference, \( d \)?

\[ a_1 = 6 \quad \text{and} \quad d = 4 \]

b. Simplify the equation completely.

\[ a_n = 6 + 4n - 4 \]
\[ a_n = 4n + 2 \]

c. What is the value of the \( y \)-intercept of this relationship?

\[ 2 \]

d. What is the value of \( a_4 \)?

\[ a_4 = 4(4) + 2 = 16 + 2 = 18 \]

For # below, write an equation using the traditional \( y = mx + b \) format and an equation using the new \( a_n = a_1 + (n - 1)d \). Simplify all equations completely and draw arrows to show the relationship between the two equations.

2. \( y \)-intercept of 2, slope of 6.

\[ y = 6x + 2 \]
\[ a_n = 6 + (n - 1)(6) \]
\[ a_n = 6n + 2 \]

3. \( a_0 = 3, d = -2 \)

\[ y = -2x + 3 \]
\[ a_n = 1 + (n - 1)(-2) \]
\[ a_n = -2n + 3 \]

4. 

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>3</td>
<td>7</td>
<td>11</td>
<td>15</td>
<td>19</td>
<td>23</td>
</tr>
</tbody>
</table>

\[ y = 4x + 3 \]
\[ a_n = 4 + (n - 1)(4) \]
\[ a_n = 4n + 3 \]
Lesson 3: Recursive Formulas

In the previous two lessons we discussed how both Michelle’s and Michael’s daily income situations were examples of arithmetic sequences. We used the general linear equation \( y = mx + b \) and the arithmetic sequence equation \( a_n = a_1 + (n - 1)d \) to represent these relationships.

### Michelle’s Daily Pay

<table>
<thead>
<tr>
<th>Lemonades Sold</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daily Pay (in dollars)</td>
<td>13</td>
<td>15.5</td>
<td>18</td>
<td>20.5</td>
<td>23</td>
</tr>
</tbody>
</table>

### Michael’s Daily Pay

- Initial daily pay of $20
- $1 earned for each iced tea sold

A. We can use NOW – NEXT statements to illustrate how to create a sequence of values recursively. The NOW variable represents the current term and the NEXT variable represents the very next term. An example is shown below.

**Sequence:** 4, 7, 10, 13, ...

\[
\text{NEXT} = \text{NOW} + 3
\]

Starting at: 4

This example says to start at a value of 4 and then add 3 to the current term in order to get the next term.

1. Write a NOW – NEXT statement for both Michelle and Michael’s daily pay. Use the \( y \)-intercept (for 0 drinks sold) as the starting value for each.

   **Michelle**
   
   \[
   \text{NEXT} = \text{NOW} + 2.5
   \]
   
   Starting at: 13

   **Michael**
   
   \[
   \text{NEXT} = \text{NOW} + 1
   \]
   
   Starting at: 20

2. Which number in each of your equations would be the slope, \( m \), if the equation was in the form \( y = mx + b \)?

   **Michelle**
   
   \( m = 2.5 \)

   **Michael**
   
   \( m = 1 \)

3. Which number in each of your equations would be the \( y \)-intercept, \( b \), if the equation was in the form \( y = mx + b \)?

   **Michelle**
   
   \( b = 13 \)

   **Michael**
   
   \( b = 20 \)
Another similar method of writing these equations recursively uses different notation to represent the same situation. An example of this notation is shown below.

**Sequence**: 4, 7, 10, 13, ...

\[ a_{n+1} = a_n + 3 \text{ for } n \geq 1 \]
\[ a_1 = 4 \]

1. Match up each part of this new formula with the previous one.
   a. NOW = \underline{a_n}
   b. NEXT = \underline{a_{n+1}}
   c. Starting at = \underline{a_1}

2. Why does this formula need to include the \( n \geq 1 \) part? Would you be able to create the sequence without this? Explain.
   "The \( n \geq 1 \) says the domain of the formula. It shows the sequence values are \( a_1, a_2, a_3, \ldots \)."

3. Rewrite your NOW − NEXT statements from A1 using this new format.
   Michelle
   \[ a_{n+1} = a_n + 2.5 \text{ for } n \geq 0 \]
   \[ a_0 = 13 \]
   Michael
   \[ a_{n+1} = a_n + 1 \text{ for } n \geq 0 \]
   \[ a_0 = 20 \]

4. What did you label your starting values in part 3 as? Explain your decision.
   \( a_0 \) \( \rightarrow \) the "first" term is when 0 drinks are sold, not 1 drink. \( a_0 \) \( \rightarrow \) initial daily pay

5. Using your formula for Michelle’s daily pay from part 4, list the steps needed to find the value of \( a_4 \).
   \[ a_0 = 13 \]
   Find \( a_1 \) by adding 2.5 to \( a_0 \) (13)
   Find \( a_2 \) by adding 2.5 to \( a_1 \) (15.5)
   Find \( a_3 \) by adding 2.5 to \( a_2 \) (18)
   Find \( a_4 \) by adding 2.5 to \( a_3 \) (20.5)
   \[ \Rightarrow a_4 = 23 \]
C. Using your work from this lesson and previous lessons, answer the following questions.

1. If you had to find the value of $a_{20}$, which formula would you use? Explain.

   **Option 1**
   \[ a_{n+1} = a_n + 3 \text{ for } n \geq 1 \]
   \[ a_1 = 5 \]

   **Option 2**
   \[ a_n = 3n + 2 \]

   Option 2. Can plug in 20 for $n$ and solve in a couple steps.
   \[ a_{20} = 3(20) + 2 \]
   \[ a_{20} = 62 \]

2. Was your choice in part 1 an explicit or a recursive relationship?

   **Explicit**

3. When is an explicit formula for an arithmetic sequence (in the form $y = mx + b$ or $a_n = a_1 + (n - 1)d$) more helpful than a recursive one?

   **Explicit** is easier to find any given sequence value without having to find previous values.

4. When is a recursive formula more useful than an explicit one?

   **Recursive** formula easily shows the starting value and the common change between values.

D. Write an explicit and a recursive equation for each situation described below.

1. Joan makes $10 per day plus an additional $1.30 for each basket she sells.

   \[ J = 10 + 1.3b \]
   \[ a_{n+1} = a_n + 1.3 \text{ for } n \geq 0 \]
   \[ a_0 = 10 \]

2. A stack of boxes at a grocery store consists of 22 boxes on first row on the bottom, 18 in the next row up, and so on forming an arithmetic sequence until the top has 2 boxes.

   \[ B = 22 + (n-1)(-4) \]
   \[ B = 22 - 4n + 4 \]
   \[ B = 26 - 4n \]

   \[ a_{n+1} = a_n - 4 \text{ for } n \geq 1 \]
   \[ a_1 = 22 \]
Extensions/Homework

1. At a large grocery store they are having a promotion to give away free samples of pasta sauce. On the first day, they start with 200 cans of sauce. They plan on giving away 12 cans of sauce per day.

   a. Write a recursive formula to represent this situation.
      \[ a_{n+1} = a_n - 12 \quad \text{for } n \geq 1 \]
      \[ a_1 = 200 \]

   b. Write an explicit formula to represent this situation.
      \[ C = 200 + (n - 1)(-12) \]
      \[ C = 200 - 12n + 12 \]
      \[ C = 212 - 12n \]

   c. Find how many cans they will have left after 8 days.
      \[ C = 212 - 12(8) \]
      \[ C = 212 - 96 \]
      \[ C = 116 \] cans

   d. Determine on which day of the promotion they will run out of sauce.
      \[ C = 0 = 212 - 12n \]
      \[ \frac{12n}{12} = \frac{212}{12} \]
      \[ n = 17.6 \]

   Write an explicit and a recursive equation for each situation described below.

2. To help study for an upcoming test, Julia decides to study each night more than she studied the previous night. She plans on studying for 10 minutes the first night and will increase by 5 minutes each night.

   \[ S = 10 + (n - 1)(5) \]
   \[ S = 10 + 5n - 5 \]
   \[ S = 5n + 5 \]

   \[ a_{n+1} = a_n + 5 \quad \text{for } n \geq 1 \]
   \[ a_1 = 10 \]

3. James is paying off his student loans every month. In the first month, his loans start at a value of $30,000. He pays them off by writing checks for $250 each month.

   \[ L = 30000 + (n - 1)(-250) \]
   \[ L = 30000 - 250n + 250 \]
   \[ L = 30250 - 250n \]

   \[ a_{n+1} = a_n - 250 \quad \text{for } n \geq 1 \]
   \[ a_1 = 30000 \]
Lesson 4: Explicit and Recursive Using Various Syntax

Just like how we can use the variables $M$ and $t$ in place of $y$ and $x$ in $y = mx + b$, we can also use different variables in place of $a_n$ and $n$ in $a_n = a_1 + (n - 1)d$.

A. Josh has a job selling tie-dye T-shirts at the beach. His pay is represented by the equation shown below:

$$a_n = 8 + (n - 1)(3)$$

1. Simplify this equation completely.

\[
\begin{align*}
a_n &= 8 + 3n - 3 \\
a_n &= 3n + 5
\end{align*}
\]

2. Rewrite the equation using different variables that better align with the context. Briefly explain why you chose your variables.

$$J = 3t + 5$$

$J$ stands for Josh's earnings
$\ t \ $ stands for number of T-shirts sold

3. Could this equation represent this situation? Explain.

Yes, it is almost identical to $J = 3t + 5$. $J_t$ just shows that $J$, Josh's earnings, are related to the number of T-shirts sold, $t$.

B. Function notation is also often used to represent a linear relationship. The form $f(x) = mx + b$ can be used in place of $y = mx + b$.

1. Rewrite your equation from A2 using function notation.

$$f(t) = 3t + 5$$

2. Explain how $a_n$ and $f(n)$ notations are alike.

Both show that $n$ is the input. The outputs are just labeled differently.

$a_1 = f(1) \rightarrow$ first term
$a_5 = f(5) \rightarrow$ fifth term
C. Recursive sequences can also be written using various forms. Two recursive formulas for Josh’s T-shirt income are shown below.

(a) \[ J_{n+1} = J_n + 3 \text{ for } n \geq 0 \]
\[ J_0 = 5 \]

(b) \[ J_n = J_{n-1} + 3 \text{ for } n \geq 1 \]
\[ J_0 = 5 \]

1. Compare these formulas. Are they both accurate ways of showing Josh’s income relationship?

Yes. They both say start with $5 for 0 shirts sold and then add 3 for each additional shirt sold.

2. Using both formulas, generate the first five terms of each sequence. Does this support or change your answer from part 1?

<table>
<thead>
<tr>
<th>( n )</th>
<th>( J_n )</th>
<th>( n )</th>
<th>( J_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>2</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
<td>3</td>
<td>14</td>
</tr>
<tr>
<td>4</td>
<td>17</td>
<td>4</td>
<td>17</td>
</tr>
</tbody>
</table>

This supports my answer to part 1 since the two sequences are the same.

3. Rewrite the equations using \( NEXT - NOW \) statements.

\[ \text{NEXT} = \text{NOW} + 3 \]
\[ \text{Starting at } 5 \]

4. 
   a. If \( J_n \) is the \( NOW \), what is the \( NEXT \)?
   \[ J_{n+1} \]

   b. If \( J_n \) is the \( NEXT \), what is the \( NOW \)?
   \[ J_{n-1} \]

5. Rewrite formula (a) using function notation like \( f(n+1) \).

\[ J(n+1) = J(n) + 3 \text{ for } n \geq 0 \]
\[ J(0) = 5 \]
D. Nicole works for the same T-shirt company as Josh and has to put up posters around town to advertise for an upcoming sale. She starts the day with 240 posters and is able to put up 3 every minute.

1. Write an explicit equation for this sequence using \( p_m \) as the number of posters left after \( m \) minutes.

\[
p_m = 240 - 3m
\]

2. Rewrite your equation from part 1 using function notation.

\[
f(m) = 240 - 3m
\]

3. Write a recursive formula for this sequence using \( p_m \) as the number of posters left after \( m \) minutes.

\[
p_{m+1} = p_m - 3 \quad \text{for } m \geq 0
\]

\[
p_0 = 240
\]

4. Rewrite your recursive formula in a different but equivalent way. Use part C as an example.

\[
p_m = p_{m-1} - 3 \quad \text{for } m \geq 1
\]

\[
p_0 = 240
\]

5. Rewrite your formula from part 3 using function notation.

\[
f(m+1) = f(m) - 3 \quad \text{for } m \geq 0
\]

\[
f(0) = 240
\]

6. Rewrite your formula from part 4 using function notation.

\[
f(m) = f(m-1) - 3 \quad \text{for } m \geq 1
\]

\[
f_0 = 240
\]

7. Out of all equations written in parts 1-6, which is the easiest one for you to understand. Justify your choice.

\#2 \( f(m) = 240 - 3m \)

Easy to see initial amount (240), change (-3), and can be used to find any term in sequence.
Extensions/Homework

1. Wood is often stacked like the picture shown at right with the bottom layer having the most and the top layer having the least. Suppose the first layer on the bottom has 48 logs. Each additional layer has 4 less logs than the layer below it.

a. Write an explicit equation for this relationship. Explain why you chose the variables you did.

\[
W_l = 48 + (l-1)(-4) \\
W_l = 48 - 4l + 4 \\
W_l = 52 - 4l
\]

b. If you answer in part a is not in function notation, rewrite it so that it is. Otherwise, rewrite it so that the equation is in \(a_n\) notation.

\[
w(l) = 52 - 4l
\]

c. Write a recursive formula for this situation.

\[
w_{l+1} = w_l - 4 \quad \text{for } l \geq 1 \\
w_1 = 48
\]

d. Write a different, yet equivalent, recursive formula for this situation.

\[
w_l = w_{l-1} - 4 \quad \text{for } l \geq 2 \\
w_1 = 48
\]

e. Using any of your formulas, find the number of logs in the first 6 layers.

<table>
<thead>
<tr>
<th>(l)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(W_l)</td>
<td>48</td>
<td>44</td>
<td>40</td>
<td>36</td>
<td>32</td>
<td>28</td>
</tr>
</tbody>
</table>
Lesson 5: Applying the Various Syntax

A. Jeff creates a plan to raise money to clean up the beach. He receives an initial donation and then receives a donation for each T-shirt sold at Josh’s T-shirt store. The store offers Jeff a few options to choose from.

For each function below, define what each variable represents. Also, determine how much the initial donation and how much the donation per T-shirt would be.

1. \( f(0) = 40 \)
   \( f(x + 1) = f(x) + 2 \) for \( x \geq 0 \)
   
   \( f(x) \rightarrow \) total in donations for \( x \) shirts
   \( x \rightarrow \) number of T-shirts sold
   \( \$2 \) per T-shirt
   \( \$40 \) initial donation

2. \( d_t = 30 + 2.25t \)
   
   \( d_t \rightarrow \) total in donations for \( t \) shirts sold
   \( t \rightarrow \) number of T-shirts sold
   \( \$2.25 \) per T-shirt
   \( \$30 \) initial donation

3. \( x_n = x_{n-1} + 2.75 \)
   \( x_0 = 25 \)
   
   \( x_n \rightarrow \) total in donations for \( n \) shirts sold
   \( n \rightarrow \) number of T-shirts sold
   \( \$2.75 \) per T-shirt
   \( \$25 \) initial donation

4. \( g(y) = 35 + 2.1y \)
   
   \( g(y) \rightarrow \) total in donations for \( y \) shirts sold
   \( y \rightarrow \) number of T-shirts sold
   \( \$2.10 \) per T-shirt
   \( \$35 \) initial donation

5. For this context, is the first term of the sequence the 0th term or the 1st term? Explain.
   
   The 0th term. In this case, 0 T-shirts could be sold so the first term is the initial donation.
B. Upon hearing about Jeff’s fundraiser, the town decides to help with cleanup on the first day. As a result, the group was able to remove 15 barrels of garbage on the first day. After that, the remaining cleaners were able to remove 4 barrels of garbage a day.

1. Complete the table below to represent this situation.

<table>
<thead>
<tr>
<th>Day</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Barrels of Garbage Removed</td>
<td>15</td>
<td>19</td>
<td>23</td>
<td>27</td>
<td>31</td>
<td>35</td>
</tr>
</tbody>
</table>

2. One of Jeff’s workers, Molly, determines that a recursive formula could represent this relationship. Describe what her formula represents.

\[ G(d + 1) = G(d) + 4 \quad \text{for} \quad d \geq 0 \]
\[ G(0) = 15 \]

Day 0 \( \rightarrow \) 15 barrels of garbage removed
Each day, 4 more barrels are collected.

3. Does her equation accurately depict this situation? If yes, explain. If no, fix her formula so it is accurate.

No. It should say \( G(1) = 15 \) since the first term is day 1. And then it would say \( d \geq 1 \) for the first part.

4. Write an explicit formula using \( g_d \) notation for this relationship.

\[ g_d = 15 + (d-1)(4) \]

or

\[ g_d = 11 + 4d \]

5. Rewrite your formula from part 4 using function, \( f(x) \), notation.

\[ f(x) = 11 + 4x \]

6. For this context, is the first term of the sequence the 0th term or the 1st term? Explain.

The 1st term. Since there is no day 0 for this situation, the first term is day 1.
C. For each arithmetic sequence defined below, do the following:

- State the first term and the common difference
- Write the first five terms of the sequence
- State whether the formula is explicit or recursive.
- If the formula is explicit, write a recursive formula. If it is recursive, write an explicit equation.
- Write the relationship as a $NEXT - NOW$ statement.

1. $f(1) = 12$
   $f(x) = f(x - 1) + 4$ for $x \geq 2$
   - First term: $12$
   - Common difference: $4$
   - Explicit: $f(x) = 12 + (x-1)(4)$
   - $f(x) = 8 + 4x$
   - $NEXT = NOW + 4$
   - $Starting \ at \ 12$

2. $b_i = 4 + (n - 1)(-2)$
   - First term: $4$
   - Common difference: $-2$
   - Recursive: $b_{i+1} = b_i - 2$ for $i \geq 1$
   - $b_1 = 4$
   - $NEXT = NOW - 2$
   - $Starting \ at \ 4$

3. $x_{n+1} = x_n - 2$ for $n \geq 1$
   $x_1 = -5$
   - First term: $-5$
   - Common difference: $-2$
   - Explicit: $x_n = -5 + (n-1)(-2)$
   - $x_n = -2n - 3$
   - $NEXT = NOW - 2$
   - $Starting \ at \ -5$
D. Throughout these problems you have seen some sequences that start with the first term as $a_1$ and others that start with $a_0$. Describe the differences between these two types of sequences. Use examples to show how each one may be used.

$a_1$ sequences have a first term when $n=1$.
Example: Garbage collected each day

$a_0$ sequences have a first term when $n=0$.
Example: Cost with an initial price ($a_0$)

Extensions/Homework
Jane decides to do her own fundraiser to help with the beach cleanup. Her results from the cleanup form an arithmetic sequence and are shown in the table.

<table>
<thead>
<tr>
<th>Day</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Barrels of Garbage Removed</td>
<td>10</td>
<td>16</td>
<td>22</td>
<td>28</td>
<td>34</td>
<td>40</td>
</tr>
</tbody>
</table>

1. What is the initial value of the sequence? What is the common difference?

\[ a_1 = 10 \]

Initial Value
\[ \text{common difference} : 6 \]

2. Write an explicit equation for this relationship. Explain why you chose the variables you did.

\[ a_n = 10 + (n-1)(6) \]
\[ a_n = 6n + 4 \]

3. Josh defines Jane’s sequence recursively as shown below. Does his sequence accurately create her results? If yes, explain. If not, what should he fix?

\[ j_{n+1} = j_n + 10 \text{ for } n \geq 1 \]
\[ j_1 = 6 \]

No, he needs to switch the 6 + 10. The first term is 10 and the common difference is 6.
ARITHMETIC SEQUENCES USING SITUATED TASKS

Summative Assessment

Multiple Choice #1-5

1. What is the common difference of the sequence $5, 2, -1, -4, ...$?
   (1) 3
   (2) -3
   (3) 5
   (4) -5

2. Which formula accurately represents the sequence from question 1?
   (1) $a_{n+1} = a_n + 3$ for $n \geq 1$
   $a_1 = 5$
   (2) $a_{n+1} = a_n - 3$ for $n \geq 1$
   $a_1 = 5$
   (3) $a_{n+1} = a_n + 5$ for $n \geq 1$
   $a_1 = 3$
   (4) $a_{n+1} = a_n + 5$ for $n \geq 1$
   $a_1 = 5$

3. What is the fifth term of the sequence described below?
   $f(n) = f(n-1) + 4$ for $n \geq 2$
   $f(1) = -3$
   (1) 13
   (2) 5
   (3) 1
   (4) 17

4. Which equation would produce the same sequence as $a_n = 2 + (n - 1)(-3)$?
   (1) $y = 2x + 3$
   (2) $y = -3x + 2$
   (3) $y = -3x - 1$
   (4) $y = -3x + 5$

5. Which explicit equation would represent the sequence defined below?
   $x_{n+1} = x_n + 3$ for $n \geq 1$
   $x_1 = 7$
   (1) $x_n = 3n + 7$
   (2) $x_n = 3n + 4$
   (3) $x_n = 7n + 3$
   (4) $x_n = 7n - 4$
   $7, 10, 13, 16, 19$
Short Answer – Show all work to support your solutions.

6. Bailey creates a pattern using blocks. In the first design, she uses 5 blocks. In the second, she uses 11 blocks. In the third, she uses 17 blocks.

a. Assuming the pattern continues in the same way. Explain why this situation represents an arithmetic sequence.

\[ \frac{11-5}{2} = 3 \quad \text{This is a common difference of 6.} \]
\[ \frac{17-11}{2} = 3 \quad \text{The domain is whole numbers since she cannot create a fractional design.} \]

b. Write an explicit equation to model this sequence.

\[ a_n = 5 + 6(n-1) \]
\[ a_n = 6n - 6 + 5 \]
\[ a_n = 6n - 1 \]

c. Write a recursive formula to model this sequence.

\[ a_{n+1} = a_n + 6 \quad \text{for } n \geq 1 \]
\[ a_1 = 5 \]

d. Determine the number of blocks Bailey would use in the 50th design.

\[ a_{50} = 6(50) - 1 \]
\[ a_{50} = 300 - 1 \]
\[ a_{50} = 299 \]

e. Which design number would need 35 blocks? Show your work to support your answer.

\[ 35 = \frac{6n-1}{6} + \frac{1}{6} \]
\[ 36 = \frac{6n}{6} \quad n = 6 \]

The 6th design
7. Josie is collecting donations for a walkathon. Her donors pledge a certain amount up front and then more for each mile she walks. Her donors provide her with the equations below to represent their pledges. For each equation, determine the initial donation, amount pledged per mile, and first five amounts for walking 1-5 miles.

a. \( f(x) = 4x - 2 \)

<table>
<thead>
<tr>
<th>Initial Donation</th>
<th>Pledge Per Mile</th>
<th>First Five Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(0) = -2 )</td>
<td>$4</td>
<td></td>
</tr>
<tr>
<td>Donation Total</td>
<td>2 6 10 14 18</td>
<td></td>
</tr>
</tbody>
</table>

b. \( b_{n+1} = b_n + 3 \) for \( n \geq 1 \)
\( b_1 = 5 \)

<table>
<thead>
<tr>
<th>Initial Donation</th>
<th>Pledge Per Mile</th>
<th>First Five Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2</td>
<td>$3</td>
<td></td>
</tr>
<tr>
<td>Donation Total</td>
<td>5 8 11 14 17</td>
<td></td>
</tr>
</tbody>
</table>

c. \( a_n = 4 + (n - 1)(3) = 3n + 1 \)

<table>
<thead>
<tr>
<th>Initial Donation</th>
<th>Pledge Per Mile</th>
<th>First Five Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1</td>
<td>$3</td>
<td></td>
</tr>
<tr>
<td>Donation Total</td>
<td>4 7 10 13 16</td>
<td></td>
</tr>
</tbody>
</table>

d. Are any of the plans from a-c unrealistic? Explain.

Plan (a) \( \Rightarrow \) initial donation of -\$2 is unrealistic since they would not donate negative dollars.
8. If $Y_n$ represents the current term of a sequence, what notation would be used to represent the next term?

\[ Y_{n+1} \]

9. If $f(n)$ represents the current term of a sequence, what notation would be used to represent the previous term?

\[ f(n-1) \]

10. Robert is planning on running a marathon of about 42 kilometers. To build up his endurance, he plans on running 1.5 km the first day and then running an additional 0.75 km each day.

   a. Write an explicit formula to represent the distance he runs each day $D_n$ after $n$ days.

   \[
   D_n = 1.5 + 0.75(n-1)
   \]

   \[
   D_n = 0.75n + 0.75
   \]

   b. Will Robert meet his goal of 42 km after a month (30 days)? If not, how close will he be?

   \[ D_{30} = 0.75(30) + 0.75 \]

   \[ D_{30} = 23.25 \]

   \[ 42 - 23.25 = 18.75 \]

   No, he is 18.75 km away from his goal.

   c. On what day will Robert reach a distance of 42 km?

   \[
   42 = 0.75n + 0.75
   \]

   \[
   -0.75
   \]

   \[
   41.25 = 0.75n
   \]

   \[
   \frac{41.25}{0.75} = n
   \]

   \[
   55 = n
   \]

   55th day