Teaching the Strand of CCSS Slope through Metacognition in Middle School, High School, and Advanced Placement Mathematics

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Teaching the Strand of CCSS Slope through Metacognition in Middle School, High School, and Advanced Placement Mathematics

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A thesis submitted to the Department of Education of The College at Brockport, State University of New York, in partial fulfillment of the requirements for the degree of

Master of Adolescent Education

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Abstract

The strand of slope presents every year from 7th grade through Advanced Placement Calculus. The Common Core Learning Standards require and encourage a thorough understanding of slope and slope-related concepts such as unit rate, parallel and perpendicular lines, and the derivative. Unfortunately, many secondary mathematics teachers struggle to teach students to monitor their process and conceptualize an overall strategy for solving complex and fundamental problems. This thesis examines the role of metacognition-focused instruction on achievement, and offers research-supported teaching methods in the context of slope and unit rate that support metacognition. These methods are then presented in a series of lessons, one each from 7th grade through calculus, examining slope through a metacognitive lens. Lessons are tied together enabling teachers to solidify their own understanding of the big picture role of slope in secondary mathematics: earlier lessons foreshadow calculus concepts, and later lessons look back to the earlier foundation of unit rate and basic uses of slope. By examining slope within a metacognitive context, teachers can strengthen standards-based curriculum, solidify understanding of slope, and incorporate teaching tools to support achievement.

Key words: mathematics, metacognition, Common Core, education, slope, derivative, unit rate, constant of proportionality, middle school, high school
Dedication

This thesis is dedicated to my children whose endless questions of “why” inspire me daily to ask good questions; to my mother the English teacher who taught me how important it is to think about thinking and articulate that thinking clearly; to my father who helped me edit my first essay on the wondrous applications of calculus in 1990; to my ever constant husband, Jeff, my greatest champion and cheerleader and APA proofreader; and to my Father in Heaven who makes all things possible.
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Curriculum Aim

Every classroom teacher’s goal is for every student to succeed. Research shows a relatively optimistic outlook in the early years; elementary students generally excel and achieve with some measure of constancy (Lee, 2012). Then come the middle school years. In 8th grade, educators see a significant slowdown in achievement (Lee, 2012). This drop coincides with the introduction of foundational concepts such as functions, slope, and irrational numbers as well as a deeper look at congruence, similarity, and linear relationships (New York State Education Department, 2013). The lack of achievement in 8th grade contributes to a domino effect with long lasting cumulative impact. This thesis offers a potentially powerful tool to help teachers mitigate this decline with the hope of producing long-lasting benefits for mathematics students.

Metacognition has long proven to provide substantial gains in achievement at all levels of mathematics (Garofalo & Lester, 1985; Goos, Galbraith, & Renshaw, 1996; Laistner, 2016; Schoenfeld, 1995). Several sources offer suggestions on techniques and strategies for teaching students metacognitive mathematical processes (Schoenfeld, 2014; Mevarech et Fridkin, 2006; Goos, Galbraith, & Renshaw, 1996). With the implementation of the Common Core Learning Standards, however, there is a real need for curriculum developed from a metacognitive framework.

Metacognition dovetails nicely with the CCLS stated goals of increased focus, coherence, and rigor (NYSED, 2012). The NYS Common Core Standards state that:

Focus in the curriculum is meant to give students an opportunity to understand concepts and practice with them in order to reach a deep and fluent understanding.

Coherence in the curriculum means progressions that span grade levels to build students’ understanding of ever more sophisticated mathematical concepts and
applications. Rigor means a combination of fluency exercises, chains of reasoning, abstract activities, and contextual activities throughout the module (NYSED, 2012).

Student awareness and monitoring of the problem solving process through metacognitive strategies can support greater focus and depth, facilitate coherence from one grade level to the next as topics are studied across multiple grade levels, and enable students to approach mathematical problem solving from a more rigorous vantage point. But how can teachers promote this awareness of these metacognitive processes?

A survey of the literature yielded no complete secondary mathematics lessons supported by research and tied to New York State standards that provide pathways to student awareness of the problem solving process through metacognitive strategies. Curricula with this focus can enhance adoption of the Mathematics Practice Standards by students (NYSED, 2013). This thesis offers one lesson from each grade from 7th through 12th (including Calculus and AP Statistics) focused on the strand of slope. Educators can benefit from curriculum that solidifies students’ conceptual development of the foundational strand of slope. This cornerstone concept is a constant challenge for students throughout every secondary mathematics course, and permeates higher mathematics as well. Poor understanding of slope in the earlier years may prevent success when students are challenged with application of slope in a pre-calculus, calculus, or statistics context. A solid middle school understanding and the continuing development of the slope concept from course to course can prepare the way for success in calculus and college mathematics. Additionally, we cannot underscore the importance of high quality mathematics instruction in the middle school years. Sadler, Sonnert, Hazari, and Tai (2012) found that students’ career interests in STEM fields are, for the most part, already
established by the time students enter high school. Among females in particular, the high school years have proven to be an unfruitful time to attract students to STEM fields, therefore STEM education in the early and middle grades is of great importance (Sadler et al., 2012). Furthermore, the strand of slope refers back to late-elementary lessons on unit rate, and looks ahead to the powerful application of the derivative in calculus. The lessons in this thesis draw on the prevalence of slope throughout secondary education and tie content together across the grades. By knowing the connections between the strand of slope and how it relates to upcoming concepts that will be taught, educators can facilitate better learning opportunities for students by hooking what they will soon need to know to what they are learning at that time.

Each lesson utilizes several methods well documented in research literature as methods that support the development of student metacognition. These lessons, in essence, provide examples of how to invite students into the culture of mathematics and help students to feel at home once they arrive. The curriculum in this thesis can be used as presented, but most importantly, it can serve as a valuable resource for educators in grades 7-12 as they (a) develop research-supported slope curriculum that solidifies metacognitive problem solving in a mathematics classroom, and (b) solidify their own understanding of the strand of slope in secondary mathematics curricula. Specifically, it aims to give teachers curriculum examples of how to develop their own metacognitively based curriculum which can head off the 8th grade achievement decline while inoculating students to the sloppy slope misunderstandings of middle school and beyond. It is the author’s hope that these lessons will broadly inform curriculum development as teachers complete the transition to the Common Core Learning Standards.
Theoretical Perspective on Mathematical Metacognition

What is mathematical learning? Sweller describes learning as a permanent change in long-term memory along with the ability to recall what has been learned (Sweller & Chandler, 1991; Sweller & Chandler, 1994; Sweller, Van Merrienböer, & Paas, 1998; Wade, 2011). Schoenfeld expanded the construct of learning in a mathematical setting to include ways of behaving and responding to mathematics and to people who use and present mathematics (1992). He proposed that metacognitive abilities compose an enormous piece of doing mathematics. But what is metacognition and how can it help student achievement? Briefly, metacognition is an awareness of thought process combined with the ability to monitor and adjust one's thought processes. Students can often become overwhelmed by the many steps and variables involved in solving a complex mathematical problem. Keeping track of these elements and how they interact creates what is known as an intrinsic cognitive load for the learner to manage. Cognitive load is dependent upon the learner’s expertise and the complexity of the task at hand (Van Merriënboer & Sweller, 2005; Wade, 2011). In a mathematical context, a student can apply metacognitive strategies to monitor and regulate their mathematical problem-solving process and lighten their cognitive load (Kalyuga, 2009).

The importance of metacognition can be illustrated in the context of group mathematics problem solving. Goos et al. (2002) found that group work can be incredibly beneficial in prompting and developing metacognitive behavior. They presented three results from examining metacognition in a collaborative context:

1) Working with a partner facilitated student articulation of ideas because “students clarified, elaborated, and justified their New Ideas for the benefit of a partner.”
2) Working with a partner encouraged students to ask for feedback and help evaluating the correctness of their work.

3) Working with a partner also required students to examine, understand, follow, discuss, and critique others’ work. (pp. 206-207)

In sum, metacognitive activity in peer collaboration happens as students offer “[their] thoughts to others for inspection, and [act] as a critic of [their] partner’s thinking (p. 207). Transactive discussion in particular facilitates metacognition “by making students’ thinking public and open to critical scrutiny” (p. 219). Essentially, it has been proposed that the social interaction with and accountability to peers created by such interaction prompts deeper thought and more effective problem solving.

Additionally, metacognitive strategies can increase the flow of ideas, and the quality of work produced (Kalyuga, 2009; Lee & Tan, 2010; Scott & Schwartz, 2007). Many students find the work in mathematics class to be challenging. There is a lot to remember, a lot to keep track of, and a lot to do when solving complex problems. Metacognitive strategies can help students keep track of all of the moving pieces and focus their attention on the steps that are most helpful at a particular moment. Heightened student awareness of where we’ve been, where we are, and where we’re going can facilitate lightened cognitive load, thereby facilitating transfer of knowledge (van Merrienböer & Sweller, 2005, p. 163). Kalyuga noted, “Metacognitive regulation determines a person’s ability to monitor, plan, evaluate, and control the processing of information relevant to the goal” (2009, p. 403).

In conclusion, metacognition as a construct clarifies internal processing of challenging mathematics problems. Research confirms the presence and benefits of
metacognitive functioning in problem solving. Research supports the use and benefit of using methods that prompt metacognitive strategies in the classroom as a tool to increase achievement. Teachers would do well to examine metacognitive frameworks as they plan curricula.

Literature Review

Metacognition and Doing Mathematics

Having touched on a few examples from research that illustrate the utility of metacognition in a mathematical context, a full literature review is presented for the teacher’s benefit. What contributes to more effective and functional metacognition in the mathematics classroom? Schoenfeld (1992) observed the resources that an individual brings to a problem-solving situation: degree of knowledge (both mathematical and metacognitive), facts, and procedures. Furthermore, "the issues related to the individual's knowledge base are: What information relevant to the mathematical situation or problem at hand does he or she possess, and how is that information accessed and used?” (p. 43).

Effective mathematicians possess highly developed self-regulation and metacognition as evidenced by the professor able to backtrack and change directions mid-problem referenced above. In his own classes, Schoenfeld (1992) effectively uses 3 questions in a college course on problem solving: “What (exactly) are you doing? (Can you describe it precisely?) Why are you doing it? (How does it fit into the solution?) How does it help you? (What will you do with the outcome when you obtain it?)” (p. 63). He acts as a coach asking these questions and prompting students to think about their process and build awareness. By the end of the semester, they have significantly developed their metacognition skills. Teaching effective problem solving and metacognition and self-regulation is very difficult to do quickly in a typical secondary classroom; the teacher’s
TEACHING THE STRAND OF CCSS SLOPE

task is challenging and complex. Students need prolonged exposure to appropriate activities. Assessments need to measure these specific problem-solving skills. (Schoenfeld, 1992).

Much research has been done on the separate topics of students’ beliefs about mathematics, and their mathematical knowledge and problem-solving abilities. The affective and cognitive domains are often considered separately. Many studies in recent decades, however, have looked at how the two interact (see Schoenfeld, 1992, for a review). Although there are many reasons for student lack of achievement, sometimes teacher practices inadvertently result in students who give up quickly and ignore context. Furthermore, teacher beliefs inform their pedagogy, which in turn can create counterproductive student beliefs. In addition, societal beliefs and cultural norms affect what society considers reasonable to expect students to be able to do. Adults in the US (when compared to Japan) are much more likely to believe that children who do mathematics easily possess an innate ability as opposed to in Japan where adults emphasize effort. The US also introduces many concepts later than in China, Japan, and Russia revealing a cultural lack of confidence in students’ mathematical abilities and many parents believe that reading (as opposed to math) needs more emphasis in curriculum. Schoenfeld summarizes what we know and what work is still to be done by tying problem solving (and of necessity, metacognition) to culture and enculturation:

"My own bias is that the key to this problem [of poorly developed metacognition and collaboration in a mathematical context] lies in the study of enculturation, of entry into the mathematical community. For the most part, people develop their sense of any serious endeavor -- be it their religious beliefs, their attitude toward music, their identities as professionals or workers, their sense of themselves as
readers (or non-readers), or their sense of mathematics -- from interactions with others” (p. 82).

This sense of mathematical culture will prove to be key to how future learners must experience mathematics. Hill, Dean, and Goffney (2007) noted: “A teacher might have strong knowledge of the content itself but weak knowledge of how students learn the content or vice versa” (p. 378). Understanding enculturation of mathematics students is a significant part of necessary teacher knowledge.

**Effective Methods for Metacognition in Teaching Mathematics**

As mentioned above, writing and collaborative problem solving can be significant pieces of effective mathematical culture in a classroom. Most mathematics classrooms, however, are complex and include many other elements, and teachers have a variety of methods and theories available to choose from. All of these pieces contribute to the sense of culture in a mathematics classroom. The research provided below presents methods and strategies that amplify student metacognitive processes in ways that contribute to a productive mathematical classroom culture. These fall under traditional teaching methods as well as mathematical cultural norms, both of which will be addressed in the curriculum.

Research on context problems and effective teaching methods was done by Kramarski, Mevarech, and Arami (2002) with 7th graders working in groups, some with metacognition training, some without. He attempted to build on “observations that both high and low achieving students struggle with such tasks: low achievers experiencing difficulty in distinguishing relevant and irrelevant information, and high achievers failing to persevere in the absence of an obvious algorithm leading to a quick solution” (as cited in Beswick, 2011, p. 377). Metacognitive training was effective with both high and low
achievers and enabled them to better tackle the process of sifting out important information and identifying the pathway to solutions.

The Dutch Realistic Mathematics Education (RME) model holds promise for assisting students with complex tasks by providing them with a framework to exercise metacognition effectively. RME offers successful strategies for using context: support the underlying mathematics, use minimal extra information, ensure there is one clear solution, and perhaps include a follow-up “safety” question to offer extra support to ensure the correct answer is achieved (Beswick, 2011, p. 383). RME also introduces students to mathematizing processes in order to “engage in guided re-invention of mathematics” (p. 383) leading to both increased understanding and appreciation of the mathematics (see also Van den Heuvel-Panhuizen, 2010). Perhaps a method such as RME could assist educators in combating common erroneous student beliefs. Schoenfeld (1992) identified many false beliefs that most students maintain about the nature of mathematics. For better or for worse, student beliefs influence how students do mathematics. These incorrect student beliefs can be tied either directly or tangentially to metacognition:

- “Mathematics problems have one and only one right answer.
- “There is only one correct way to solve any mathematics problem -- usually the rule the teacher has most recently demonstrated to the class.
- “Ordinary students cannot expect to understand mathematics; they expect simply to memorize it, and apply what they have learned mechanically and without understanding.
- “Mathematics is a solitary activity, done by individuals in isolation.
• “Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less.

• “Formal proof is irrelevant to processes of discovery or invention” (p. 69).

Aligning student beliefs with the true nature of mathematics will create a healthier mathematical culture in the classroom and facilitate student success. In addition to direct metacognitive training, methods such as RME show promise for combating negative cultural impediments to student success.

Other researchers (Ainley, Pratt & Hansen, 2006) have identified effective components of situated views. Using authentic problems and working from a constructionist point of view “are inherently purposeful” for students and can remove some obstacles to effective metacognition (Beswick, 2011, p. 378). Beswick also cites research (Ainley et al., 2006; Boaler, 1993) suggesting “students be given sufficient autonomy in their approach to a task that they can construct their own goals for the activity” (p. 378). Teachers must take care to identify students’ perspectives and work to align them with teacher objectives. Further research rejects “the traditional approach of learning procedures and abstract ideas first and only considering applications later” in favor of teaching “mathematical ideas in contexts [which] can facilitate the development of understanding of them” (p. 379; see also Ainley et al., 2006; Boaler, 1993; Freudenthal, 1968). Jurdak (2006) found that when comparing “problem solving in the real world with problem solving conducted in a school context but intended to replicate a real-world situation,” he found that students viewed these two activities as “fundamentally different, involving different goals, offering different tools and operating under differing conditions with respect to rules and division of labour,”
suggesting opportunities for both cognitive and metacognitive discussions surrounding context problems (as cited in Beswick, 2011, p. 380).

Significant research has been done on problem solving in the context of metacognition. Year 12 students were found to successfully monitor their levels of effort and maintain focus on the task. They were found to use “‘self-talk’ to provide affirmations and to overcome the temptation to give up in the face of difficulties” (Beswick, 2011, p. 380). Beswick wonders if this persistence and facility in metacognition was developed because of contextualized problems, or more likely, as a result of exposure to increasingly difficult problems requiring more time to solve and obstacles to overcome (p. 380-381). We also know from Edwards’ work (2007) that students benefit from friendly long-term peer working relationships; perhaps these types of relationships facilitate more productive metacognition. Beswick outlines some of the difficulties that arise in researching effects of using context problems on student affect including pedagogy inconsistencies and student. Relatively speaking, little research has been done into context problems and student affect. Much is yet to be learned, and it is incumbent upon teachers to figure out how to overcome the resulting “inequities in motivation” that occur when, for example, "many students are motivated to engage with mathematics for utilitarian ends of their own (e. g. entry to university or advantage in the employment market), many others are not” (p. 381). Classroom teachers will quickly recognize this dilemma in their own schools.

Educators and researchers alike know that “. . . meaningful mathematics learning occurs in the process of engaging and grappling with not yet understood mathematics, rather than by learning procedures and then attempting to apply them (Beswick, 2011, p. 382; see also Ainley et al., 2006; Boaler, 1993; Van den Heuvel-Panhuizen, 1999).
Beswick suggests that instead of asking whether problems are contextualized or de-contextualized, educators instead should find value in using problems that offer significant challenge, and that introduce students to the purpose and utility of the mathematics they are learning (p. 382; see also Ainley et al., 2006). Utility can facilitate engagement, but Ainley et al. (2006) and Cobb (2007) promote doing mathematics itself as a worthy goal and caution against always requiring that mathematics have a use outside of itself. The curricula in this project employs examples of both types of problems: contextualized and decontextualized.

As seen above, survey of the literature yields many sources for incorporating metacognitive techniques into teaching. Furthermore, the benefits of metacognition for mathematical problem solving are well established. Since the adoption of the Common Core Learning Standards across the United States, however, schools have scrambled to find appropriate curriculum. In New York State, teachers lament the scarcity of high quality Core-aligned materials (source). This author’s survey of existing research and curriculum found no curricula explicitly teaching CCLS-aligned slope-related topics from a metacognitive perspective. This thesis will provide examples of such curricula.

**Curriculum Method**

The curriculum itself offers lesson plans (see Appendix B), which include specific applications of metacognitive instruction, suggestions on developing mathematical norms in the classroom that support metacognition, as well as sample assignments and assessments. To summarize the above research, the curricula draw from the following elements:

- Use of probing questions in both written and spoken mathematics in the classroom such as “What (exactly) are you doing? (Can you describe it
precisely?) Why are you doing it? (How does it fit into the solution?) How does it help you? (What will you do with the outcome when you obtain it?)” (Schoenfeld, 1992)

- Written responses that invite inspection of the problem solving process (Connolly & Vilardi, 1989; Countryman, 1992; Maimon, Nodine, & O’Connor, 1989; Pugalee, 2001)
- Teacher modeling of metacognitive thought processes and metacognitive language (Laistner, 2016)
- Group work (including requiring students to articulate their process, ask for feedback, and critique each others’ work (Goos et al., 2002)
- Positive feedback and acknowledgement of failure as a productive and important part of mathematical problem solving (Sweller, 2009)
- Using context problems (Ainley et al., 2006; Boaler, 1993; Freudenthal, 1968)
- Discussion of common misconceptions about doing mathematics (it is collaborative and social, and often circuitous (Schoenfeld, 1992)
- Ensuring student understanding of the goals and benefits of the above methods inviting an awareness of thought and process (Schoenfeld, 1992).
- Written work including appropriate amounts of extraneous information so as to not overload cognitive load (Beswick, 2011)
- Using problems that offer significant challenge, and that introduce students to the purpose and utility of the mathematics they are learning (Ainley et al., 2006; Beswick, 2011; Cobb, 2007)
- Analysis of worked problems requiring articulation of evidence of mathematical thinking (van Merrienböer et al., 2005)
Acknowledgement throughout the school year that mathematics problems can be challenging and time consuming, require persistence, is often collaborative and circuitous, can have more than one right answer, and can often be solved in multiple ways (Schoenfeld, 1992)

Other methods are used as appropriate. Each lesson includes a section, which shines a spotlight on one (or more) of the methods listed above.

Several of the methods listed above fall under the umbrella of teacher demeanor. Teachers can support student metacognitive function by adopting attitudes, habits, and patterns that facilitate and promote student metacognitive processes. Classrooms where discussion and collaboration as norms promote metacognition. Students need to feel comfortable taking risks and thinking out loud while attempting to solve problems. The classroom that rewards only correct and complete solutions can discourage the risk taking and circuitous collaboration that are so central to successful metacognition in mathematics. Teachers should be aware that their attitudes would set the tone for student work. By modeling thinking out loud teachers can be an example of how to “do mathematics” successfully. Success should be defined in such a way that metacognition is rewarded. The process of exploring potential routes to a solution is just as important as rewarding that correct answer. Teachers do sometimes make mistakes themselves, and they can use those rare occurrences to model error analysis and course correction. Failure to find a correct solution should be seen as success at eliminating a route and an opportunity to find a correct solution eventually. Of importance is student understanding of the goals and benefits of the above methods inviting an awareness of thought and process (Schoenfeld, 1992). Wise teachers explicitly shine light on why chosen activities and methods are beneficial and bring awareness to students of metacognitive processes
and methods (Schoenfeld, 1992). The above teaching methods were chosen because research confirms that they support metacognition in mathematical problem solving which in turn supports achievement and improves learning through permanent changes in long-term memory.

All lessons are aligned to the CCLS and Mathematical Practice Standards as listed below [complete these pieces]:

1. Make sense of problems and persevere in solving them.

2. Reason abstractly and quantitatively.

3. Construct viable arguments and critique the reasoning of others.

4. Model with mathematics.

5. Use appropriate tools strategically.

6. Attend to precision.

7. Look for and make use of structure.

8. Look for and express regularity in repeated reasoning. (NYSED, 2013)

Additionally, the expanded definitions of the Mathematical Practice Standards from the above source are included in Appendix A for teacher reference. All statements of NYS Common Core Standards are taken directly from State Department of Education resources (NYSED, 2012) but for ease of reading and formatting, standards themselves as referenced in the lessons are not in quotation marks. The following lessons have been developed for a 40-minute math class that meets daily. Lessons can obviously be adapted and utilized in various different settings. Experienced mathematics teachers have reviewed this curriculum, but it was not field tested.
Lessons

Lesson 1: Grade 7

Big Idea: Unit Rate

Focus Question: How much does ONE of something cost? How far can we travel in ONE hour? How many pencils does ONE person need? How many students in ONE class?

NYS Common Core Standards:
Ratios & Proportional Relationships 7.RP
Analyze proportional relationships and use them to solve real-world and mathematical problems.
7.RP.1 Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person walks 1/2 mile in each 1/4 hour, compute the unit rate as the complex fraction \( \frac{1/2}{1/4} \) miles per hour, equivalently 2 miles per hour.
7.RP.2 Recognize and represent proportional relationships between quantities.
   a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin. [We’ll assume for our purposes that this was completed in a previous lesson.]
   b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
   c. Represent proportional relationships by equations. For example, if total cost \( t \) is proportional to the number \( n \) of items purchased at a constant price \( p \), the relationship between the total cost and the number of items can be expressed as \( t = pn \).
   d. Explain what a point \((x,y)\) on the graph of a proportional relationship means in terms of the situation, with special attention to the points \((0, 0)\) and \((1, r)\) where \( r \) is the unit rate.

Mathematical Practice Standards:
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
8. Look for and express regularity in repeated reasoning.

Objectives:
Students will know: the definition of a unit rate.
Students will be able to: identify and find a unit rate given a table, graph, or verbal description of a proportional relationship. Students will be able to interpret the meaning of a unit rate.

Assessment/Evidence:
Performance Tasks: Complete worksheet in groups. Students will complete the assigned homework.
Other Evidence: Complete post-it note summary activity.

Although students traditionally begin learning about the concept of slope in 8th grade, we begin our lessons in this thesis in 7th grade to illustrate the importance of laying the groundwork. Metacognitive strategies taught in the earlier grades will enable students to think rigorously about complex ideas such as slope as the material becomes increasingly challenging. Students are introduced to the concept of unit rate in 6th grade in curriculum aligned to the NYS Common Core Standards. In 7th grade, the concept of unit rate enables us to help students understand the connection between a rate of change, a constant in an equation, and the steepness of a line.

Long before students struggle with advanced topics of high school courses, teachers should introduce mathematics using metacognitive strategies. Although the elementary grades are beyond the scope of this thesis, this author recommends that even, and perhaps especially, toddlers and preschoolers will respond to the question “Why?” By inviting students to articulate their reasoning and explain the mathematics that they are doing, teachers support metacognitive processes and encourage deeper thinking (Schoenfeld, 1992). In this particular lesson, teachers can ask students probing questions such as “Why did you solve the problem that way?” or “Why did you take this particular step here?” Schoenfeld also offers several powerful probing questions as discussed above. In this particular lesson, teachers can clarify the context of real-life problems by asking questions such as “Why do you think we are focusing on ONE of everything today?” and “What is the difference between miles per gallon and gallons per mile?” Teachers also should model metacognitive thought processes and language (Laistner, 2016). Whenever working a problem, either in front of the entire class or for one student, teachers can model metacognition by thinking out loud. Consider explaining the end goal
and potential steps to get there before beginning to work a problem. Better yet, ask a student to do so. Mention the steps as you progress through them while solving the problem. Once a solution is found, reflect back on the steps taken to arrive at the solution. In this lesson, our first question models this process.

We begin the lesson by asking a question about a topic that students are genuinely interested in: If I buy a bag of miniature candy bars to use as prizes in our class, and the bag contains, 300 pieces, how many pieces (on average) could ONE person receive? (This would be particularly meaningful in a class where the teacher offers candy from time to time!) Let us suppose that there are 19 students in the class. Students will be able to fairly quickly arrive at a number, 15.8. This question serves to provide a context for discussion. The teacher should ask the class why they divided 300 by 19? Why not divide 19 by 300? Why divide and not multiply, add, or subtract? Our answer is 15.8 but what is our unit? Prompt the class to assign units to 300, 19, and the answer of 15.8. The teacher should emphasize the use of the word “per” to signal to students which quantity will be divided by another. Teachers may also emphasize that this lesson offers another application of a constant of proportionality, a concept that students may already be familiar with. All problems on the worksheet for this lesson in the appendix (except for number 5) can be viewed as questions about the constant of proportionality. Teachers may wish to ask students why this concept is not applicable to question number 5.

Next, we present several questions that introduce students to unit rate in various contexts. These are contained in the Appendix as a class worksheet. We suggest that the teacher work perhaps one or two more problems with the class as guided practice as needed, and then allow students to work in pre-arranged small groups, where each student has been assigned a number, one problem at a time. Each group should discuss
their process and their answer with the aim of having every member of the group able to articulate what the group decided. The teacher will then call a number and each group should have a person assigned this number. These few students will present their group’s solution to the class for discussion. This technique, sometimes referred to as “Numbered Heads,” will encourage all students to participate in group-discussions and strive to understand each question. By discussing their thinking with each other, students can activate metacognitive processes that can support learning. The teacher may transition to independent group practice as students are demonstrating adequate understanding. As students work together in groups, the teacher should circulate the room listening, asking probing questions, restating student’s concepts, and modeling metacognitive activities.

We conclude our lesson with two brief activities. First, the teacher will conduct a Post-it summary on the question “what is a unit rate?” Each student will write their answer to this question on a post-it note, and then stick the note on the board. The teacher will read off several examples to summarize and reinforce the definition of a unit rate. The teacher will then require students to add the class definition of unit rate to their notes. This should be added to a place where students will be sure to have it saved. If the teacher has students keep “notes” separate from “worksheets” or “class work” then a definition of the unit rate should be added to formal notes. During these years of organizational growth, notes may be less likely to be lost than worksheets sometimes are. One suitable definition follows, although teachers may adapt this to fit their particular curriculum: A unit rate is a comparison of two numbers in ratio form where the second number is one.

Please note that Standard 7.RP.2d is not included in the classroom lesson; however, we recommend that the homework assignment include an anticipatory exercise
setting the stage for further work on this standard the following day. This will give them more exposure to the frequent application of a slope in the form of a fraction with a denominator of 1. Work on this standard will prepare students for 8th grade work using the slope definition, the traditional $\frac{\text{rise}}{\text{run}}$ conceptualization of slope, and the convention of converting whole number slopes into a fraction with a denominator of one. Additionally, we assume that classes would spend more than one day covering the standards listed at the beginning of this lesson.
Lesson 2: Grade 8

Big Idea: Discover the Concept of Slope (definition of slope)

Focus Question: How can one measure the steepness of a line?

NYS Common Core Standards:
Expressions & Equations 8.EE
Understand the connections between proportional relationships, lines, and linear equations.
8.EE.5 Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.

Mathematical Practice Standards:
MP1. Make sense of problems and persevere in solving them.
MP2. Reason abstractly and quantitatively.
MP3. Construct viable arguments and critique the reasoning of others.
MP5. Use appropriate tools strategically.
MP6. Attend to precision.
MP8. Look for and express regularity in repeated reasoning.

Objectives:
Students will know: the definition of slope and how it applies in the context of a graph or equation.
Students will be able to: find the slope of a line given two points, a graph of a line, a table, or an equation of a line.

Assessment/Evidence:
Performance Tasks: Students will complete the inquiry worksheet leading to a discovery of the concept and definition of slope.
Other Evidence: Students will complete the exit ticket reflecting on the definition of slope.

We will assume that teachers spend multiple days developing understanding of the above standard, as slope is a rich and nuanced concept. This lesson formally introduces the slope concept for the first time. Because the concept of slope is powerful, rich in application, and appears in rigorous and significant mathematics throughout high school and beyond, it is imperative that teachers introduce students to the big picture, albeit briefly. We do this by working with various graphs of real life situations beginning with straight lines, and ending with a quick look at complex curves. All along the way,
we examine what change (or slope) would mean in various contexts suggesting to students the power of the mathematics that they are beginning to learn. By introducing students to the end goal of the mathematics they are learning, teachers enrich student metacognition (Ainley et al., 2006; Cobb, 2007). Furthermore, slope provides rich ground for context problems, which facilitate engagement and awareness of purpose (Ainley et al., 2006; Boaler, 1993; Freudenthal, 1968). This lesson includes such problems. Additionally, we recommend showing an excellent video from PBS that introduces a very practical application of slope in the real world to increase engagement.

In the middle school and early high school years, students often still struggle with negatives and sign changes. This is an excellent topic for error analysis of worked problems requiring articulation of evidence of mathematical thinking (van Merrienböer et al., 2005). By examining incorrectly worked problems requiring use of the slope formula, students develop a greater awareness of the need to be careful with their signs. They also develop an eye for catching their own mistakes. As students are often still mastering basic arithmetic skills, it is important that written work including appropriate amounts of extraneous information so as to not overload cognitive load (Beswick, 2011). The problems below fall under this umbrella. Lastly, this lesson requires a reminder that positive feedback and acknowledgement of failure are productive and important parts of mathematical problem solving (Sweller, 2009; Wade, 2007). As students find and correct mistakes, the wise classroom teacher will offer encouragement and comment on the benefits and lessons of failure. Mistakes will be an opportunity to learn, try again, and find success.

Our lesson begins with an introductory activity where students recall past experience with steepness of lines. We lead students through review and simultaneously
set the stage for our definition of slope through an inquiry activity included in the appendix. The first two problems can be used as warm-up questions with the rest of the inquiry activity saved for later in the lesson. When discussing these questions with the class, it is important to consider the “unit rate” as a “rate of change.” Using this particular vocabulary will facilitate students’ connections between 6th and 7th grade concepts with a solid understanding of the new concept of slope. Additionally, the worksheets require that students draw a right triangle between two points on a line to aid in the calculation of slope. Teachers may wish to provide additional directions on this technique to ensure that students accurately calculate the horizontal and vertical change from one point to the next. This technique provides scaffolding for students enabling them to keep track of their thought process while calculating a rate of change. Next, we use a PBS video, Slope and House Construction, to introduce utility of slope in real life applications (PBS, 2017). Discuss possible other applications of the slope context. Concrete examples could include the incline of a handicapped access ramp or a skateboarding ramp, a ski slope or sledding hill, a hiking trail, a railing along a staircase (this could provide an anticipatory conversation about slope and parallel lines), and so on. A word of warning is included here: the word “slope” has a very specific definition in mathematics and the above-mentioned video uses a different definition. If teachers use this video, it is important to differentiate clearly between slope as used in the video and the formal definition of slope as used in mathematics.

We next recommend dividing the class into groups of 3-4 students. Each group should have access to a SMART Board, white board, or chalkboard with a coordinate plane in place to work on. Give each group a different pair of points with all groups working with pairs of points from the same line. The teacher will explain that we are now
examining lines that do not pass through the origin. We can still examine the steepness of the line by considering how much the line changes vertically and horizontally from one point to another. As a class, examine several lines each from several perspectives using different points on the same line and then compare ratios to see if there are any patterns. Essentially, the teacher will have the class calculate the slope of the same line using several different pairs of points. We recommend that the teacher demonstrate the goal using the first set of points. Plot the points, sketch the line, and count the horizontal change *emphasizing that we will go from left to right.* Then count the vertical change going up. The desired number is the ratio of these two, colloquially known as \( \frac{\text{rise}}{\text{run}} \). This is also an excellent time to pose several questions for students to think about while graphing and performing their calculations, perhaps posting these on the SMART Board:

- Does it matter which points we use to calculate this ratio? If we all use the same line but different points, will we get the same ratio? Why or why not? Give all students several minutes to complete their activity. Compare results. All groups should have obtained the same ratio of vertical to horizontal change. We trust that teachers will be able to choose lines and points that meet the ability level of their students. The goal is to get students talking together, plotting points, finding ratios, and discussing the above questions. By articulating their reasoning, metacognitive skills can be activated.

Following the above group activity examining ratios and slant of lines, the inquiry activity in the appendix can be completed with students working in small groups. Students will examine application of the calculations they just completed and further solidify their understanding of slope. At the end of the lesson, we recommend that teachers elicit summary statements from the class for consideration. Again, reinforce that slope can be interpreted consistently as a “rate of change” and often as a “unit rate” to
place the new concept of slope within a familiar context for students. Discuss the questions posed before the above activity. Follow discussion by introducing the formal slope definition. Space is provided on the inquiry worksheet for students to record a formal definition. Define the variables, work through several calculation examples simply finding slope, and then provide guided and independent practice. As this is more traditional rote practice, we will assume that teachers have access to or can create their own exercises. Further practice for homework is recommended as needed. We offer two brief activities to close the lesson. First, we suggest an error analysis to draw student attention to common difficulties with negatives in the slope formula. A sample activity is included in the appendix. Lastly, we recommend that the teacher offer students a taste of the power of the slope concept by showing and discussing with the class the value of knowing the slope at a particular point along the graph of a function. Several possible graphs are included in the appendix for discussion. These graphs are real-life examples and are intended to spark student curiosity and answer that ever-present question: “When will I ever use this?”
Lesson 3: Algebra 1

**Big Idea:** Point-Slope Equation of a Line

**Focus Question:** How can we find the equation of a line if we are given the slope and a point on a line that is NOT the y-intercept?

**NYS Common Core Standards:**

Creating Equations

A-CED

Create equations that describe numbers or relationships.

1. Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

S-ID.7 Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

MP.8 Look for and express regularity in repeated reasoning.

A-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

**Mathematical Practice Standards:**

MP1. Make sense of problems and persevere in solving them.

MP2. Reason abstractly and quantitatively.

MP5. Use appropriate tools strategically.

MP6. Attend to precision.

MP7. Look for and make use of structure.

MP8. Look for and express regularity in repeated reasoning.

**Objectives:**

**Students will know:** the point-slope form of an equation of a line and how it relates to both the slope formula and the slope-intercept equation of a line.

**Students will be able to:** find the equation of a line given the slope and a point (not the y-intercept) on the line. Students will be able to explain the meaning of slope in a given context.

**Assessment/Evidence:**

**Performance Tasks:** Students will calculate slope of a line, and find the equation of the line in both slope-intercept and point-slope form given a table of values.

**Other Evidence:** Students will write in their journals about slope. Specifically they will explain the relationship between the slope formula and the point-slope form of an equation of a line. Complete K-W-L Graphic Organizer.

The point-slope form of an equation of a line appears nowhere in the NYS

Common Core Standards or in the Common Core Standards as presented by the Council of Chief State School Officers (CCSSO) and the National Governors Association Center
for Best Practices (NGA Center). Despite the lack of explicit inclusion, most teachers include the point-slope form as it most certainly has an important role in secondary mathematics education. Several standards (listed above) imply its inclusion. More importantly, understanding of the point-slope form foreshadows several important pre-calculus and calculus concepts. The point-slope form parallels the horizontal and vertical shifts of functions that feature so prominently in pre-calculus curricula. The limit definition of the derivative (as outlined in lesson 7 of this thesis) builds on the point-slope form. The tangent line is often very effectively expressed in point-slope form. We strongly advocate the inclusion of the point-slope form as a building block in student understanding of the concept of slope.

This lesson introduces the point-slope form of the equation of a straight line by deriving the point-slope form from the equation for slope given two points. By couching point-slope form in the context of the most fundamental definition of slope that students have thus far seen, students can better understand the meaning of the point-slope form, remember it better, and be able to connect it to their intuitive understanding of change and slope.

In this lesson, we will use a K-W-L Graphic Organizer. This activity prompts students to review background knowledge (K), and wonder (W) about further implications of what they have already learned at the beginning of a lesson. Then, at the end of the lesson, students return to the graphic organizer to summarize what they have learned (L). We will assume that this is a regular activity in the classroom and that students are familiar with the format and instructions. The purpose of a K-W-L Graphic Organizer is to activate background knowledge and provide students with a firm foundation on which to expand their knowledge. At the conclusion of the lesson, the K-
W-L chart allows students to synthesize new learning. These writing activities invite metacognition. Students are thinking about what they have learned, what they may learn next, and then what they have learned in this new context. This is an excellent context for providing explicit metacognitive instruction. Teachers can explain what metacognition is: thinking about thinking. Students can learn to pay attention to where they have been, where they need to go, and where they are at any point in that journey. Teachers should draw explicit attention to these processes and encourage students’ attention to metacognition. In addition, writing about mathematics prompts metacognitive thought processes and provides a scaffold for student learning and facilitates complex thinking.

The author recommends beginning this lesson with the K-W-L Chart contained in the Appendix. This can be passed out or perhaps placed on a SMART Board as a warm-up activity. This chart will remind students of the process for finding the equation of a line using the slope and y-intercept. Once students have filled out the chart, the teacher can lead a discussion sharing ideas, perhaps drawing names to encourage all students to stay engaged. The class can then prepare a whole-class version of the K-W-L Chart to consolidate students’ ideas.

This activity leads naturally to a question that ensures student understanding of our goal: How can we find the equation of a line if we are given the slope and a point on a line that is NOT the y-intercept? We recommend a brief discussion-based notes session. The teacher will lead a discussion and act as scribe as the class together derives the point-slope formula for a line from the slope equation. Providing students with probing questions and information about the goal of the mathematics being done prompts metacognition and awareness of the utility of mathematics (Schoenfeld, 1992). A suggested format follows.
The author proposes that teachers begin their lesson by asking students for an equation of a line containing the point \((4, 2)\) with a slope \(m = -2\). The teacher should allow sufficient time for students to come to the conclusion that information is lacking. The class does not yet have enough information to use the slope-intercept form a line, as students have presumably previously found linear equations. Throughout this discussion, teachers can model metacognitive thinking out loud with statements such as: “What do we know? Do we have enough information to write the equation of the line? What else do we need to know? Are there other relationships that we could use?” Rich discussion can accompany these types of questions and observations. Discussion should lead toward the slope formula and we will examine the tradition formula for finding the slope between two points that expresses the ratio of the change in \(y\)-values to the change in \(x\)-values,

\[ m = \frac{y_2 - y_1}{x_2 - x_1}. \]

The teacher should ask the class what they recommend using instead of the coordinates of a second point. Inserting the given information yields the following: \(-2 = \frac{2 - y_1}{4 - x_1}\). Next, the teacher can ask the students what the point \((x_1, y_2)\) represents. Following several comments, the teacher should help the class conclude that this point could represent any point on the given line. At this point, the teacher can again reiterate that this equation could represent any point on the given line. At this point, to justify our next step in the derivation of the point-slope formula, the teacher could ask if anyone likes fractions. This author suspects that the class will agree to multiply both sides of the above equation by the quantity \((4-x_1)\) facilitating a revision of the above equation: \(-2(4 - x_1) = (2 - y_1)\). Next, the teacher should inquire as to how this line expresses the original information given in at the beginning of the discussion. Students will identify the slope of -2 and the presence of the coordinates of the given point, \((4,2)\),
in the equation. At this point the teacher can reveal that this equation is very close to what we call the point-slope form of the equation of a straight line. The teacher can ask if this process could have worked with a different point? With any point? Suppose we replace the point (4, 2) with a generic point \((x, y)\) and the slope -2 with \(m\). Then we have \(m(x - x_1) = (y - y_1)\). At this point, the class has essentially derived the point-slope formula of a line. This is an excellent point for the teacher to explain that this is an alternative form for the equation of a straight line to be used when we have different information from what students are accustomed to, namely slope and \(y\)-intercept. The teacher should point out that this is often presented in reverse order as \((y - y_1) = m(x - x_1)\). We recommend that students now pair up and discuss the connection between the point-slope equation of a line and the slope formula. What is the connection? What does the point-slope equation represent?

The teacher should now refer to the original question and pose a follow up question. If we are given a different slope of 3 and a different point of (-7, -2), can we use our new point-slope to quickly find an equation of a line? Students can pair up, substitute the given information into the point-slope form, and find an equation. By then going over the question as a class, perhaps having a student offer and model the solution, the class has an opportunity to explore the role of subtraction with negative numbers in the point-slope form. Errors with negative numbers are common with the point-slope form. By reviewing this together, students have an opportunity to analyze and correct errors thereby promoting metacognition.

At the conclusion of the lesson, students should refer again to the K-W-L Chart from the beginning of the lesson and complete the last part. This is an opportunity to articulate their learning through writing thereby activating metacognitive processes.
Standard A-REI.A.1 specifically requires students to be able to explain and articulate their reasoning and this is an important piece of metacognition and successful metacognition. Writing about their reasoning will support this standard. Exercises should follow either in class or as homework allowing students to gain experience using the point-slope formula in traditional ways. Since there are many ways to explain the concepts covered on the K-W-L Chart and it is mainly a tool for student reflection, no key is provided in the Appendix.
Lesson 4: Geometry

Big Idea: Parallel and Perpendicular Lines

Focus Question: Given two lines, are they parallel, perpendicular, or neither?

NYS Common Core Standards:
G-GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

Mathematical Practice Standards:
MP1. Make sense of problems and persevere in solving them.
MP5. Use appropriate tools strategically.
MP6. Attend to precision.
MP8. Look for and express regularity in repeated reasoning.

Objectives:
Students will know: the meaning of parallel and perpendicular.
Students will be able to: Students will be able to identify parallel and perpendicular lines. Students will be able to identify the slope of a line, and use that to create an equation for a new line parallel or perpendicular to the original given line.

Assessment/Evidence:
Performance Tasks: Students will complete the inquiry worksheet in groups. Students will practice identifying the slope of a second line parallel or perpendicular to a given line, and writing an equation in point-slope line.
Other Evidence: Students will participate in the active identification of parallel or perpendicular lines, or neither.

In this lesson, students expand their understanding of slope to include identifying parallel and perpendicular lines. We develop this understanding through inquiry in the context of group work offering opportunities to reflect and critique ideas through discussion. By concluding with application problems, students are able to solidify their understanding of the construct of slope as unit rate. The inquiry worksheet and exit ticket discussed below are included in the Appendix.

It is recommended that teachers begin class time with a warm-up question reviewing how to graph lines given an equation in slope-intercept form: $y = 2x + 1$ and $y = 2x - 3$. This pair of equations will also serve as an anticipatory set providing a visual
for the question: “Given the equations of two lines, can you tell if two lines are parallel or perpendicular?” As this is an anticipatory question, a firm answer is not required but discussion should elicit thoughtful responses building on background knowledge and understanding about the concepts of parallel and perpendicular. Students may see the relationship between slope and parallel lines at this point and notice that parallel lines have the same slope.

Students will now break up into small groups to complete the inquiry worksheet with discussion questions. The worksheet invites students to graph several sets of equations. In the first section, each pair of equations results in the graph of two parallel lines. Equations are given with varied slopes: positive and negative numbers, as well as fractional slopes. By the end of the section, students should have had enough experience to notice that when the slope is the same, the lines will be parallel. Students are then prompted to articulate their conclusions through writing and in discussion. After discussing their conclusions, they are prompted to revise their writing to reflect group consensus. A similar set of exercises, discussion, and writing follows for perpendicular lines. The class can regroup and discuss as a whole to ensure consistent understanding throughout the class. If time allows, the teacher may wish to provide class work or perhaps a white board quiz on quick identification of slopes for parallel and perpendicular lines, as well as writing equations of lines in point-slope form. An exit ticket is included for use at the end of the lesson that invites students to identify pairs of equations as parallel or perpendicular, and to describe the relationship between slope, and parallel and perpendicular lines.
Lesson 5: Algebra 2

**Big Idea:** linear modeling with unit rate

**Focus Question:** If we know sometime about today, can we make predictions about the future? Can we tell the future??

**NYS Common Core Standards:**
A-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
A-REI.C.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables. (Shared with A1)
F-IF.B.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. (Shared with A1)
F-LE.B.5 Interpret the parameters in a linear or exponential function in terms of a context. (Shared with A1)

**Quantities N-Q**
Reason quantitatively and use units to solve problems.
1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.
2. Define appropriate quantities for the purpose of descriptive modeling.

**Creating Equations A-CED**
Create equations that describe numbers or relationships.
1. Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

**Interpreting Categorical & Quantitative Data S-ID**
Summarize, represent, and interpret data on two categorical and quantitative variables
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.
   a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.
   c. Fit a linear function for a scatter plot that suggests a linear association.

**Interpret linear models**
7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.
8. Compute (using technology) and interpret the correlation coefficient of a linear fit.

**Making Inferences & Justifying Conclusions**
Understand and evaluate random processes underlying statistical experiments
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?
Mathematical Practice Standards:
MP1. Make sense of problems and persevere in solving them.
MP2. Reason abstractly and quantitatively.
MP4. Model with mathematics.
MP5. Use appropriate tools strategically.
MP7. Look for and make use of structure.

Objectives:
Students will know: the meaning and significance of a calculator’s Linear Regression information.
Students will be able to: find, compare, and analyze the linear regression lines (specifically slope and y-intercept) for two sets of data.

Assessment/Evidence:
Performance Tasks: Students will complete the inquiry worksheet.
Other Evidence: Teacher observation and exit ticket.

We assume that students have already been introduced to the mechanics of finding a linear regression line given a set of data on a graphing calculator. This lesson deepens understanding of interpreting those results. We also point out that this material could be an opportunity to implement the flipped classroom model. Research supports flipped classrooms for increasing student achievement in STEM classrooms including mathematics (Berrett, 2012; Fulton, 2012; Love, Hodge, Grandgenett, & Swift, 2014). This may be because students are given greater opportunity to engage in activities in class that stimulate metacognition such as group work, discussion, and problems so challenging that students would have difficulty attacking them at home independently. Additionally, metacognitive teaching strategies include teacher modeling and probing questions from the teacher and a flipped classroom allows more flexibility to incorporate these types of teacher modeling and facilitating as opposed to spending time on traditional teacher-centered lecture. In this lesson, students are guided through several examples on an inquiry worksheet found in the Appendix. In each example, students are given two sets of real-life data and asked to find the linear regression lines. We have
attempted to make these real-life problems authentic and genuine in an effort to increase engagement and stimulate metacognition. Students then interpret the slope and y-intercept of these lines with particular focus on unit rate. This lesson looks backward to concepts of unit rate introduced in 6th grade and developed in the intervening years. We look forward to Calculus and AP Statistics through discussion and analysis. Examples include discussion of the slope of a tangent line suggesting the derivative in Calculus. Students also have an opportunity to think briefly about analysis of nonlinear data sets such as those encountered in AP Statistics. The students are asked to find, interpret, and compare r-values. The CCLS require students to understand and evaluate correlation coefficients. An optional extension beyond the CCLS would be to work with $r^2$ and we leave that to individual teacher discretion depending upon student readiness.

Additionally, please note that some schools use $a + bx$ for regression lines while others use $ax + b$ for regression. Teachers can adapt worksheets to align with their own school and district conventions and curriculum.

As this lesson is centered on authentic application of mathematics, we encourage teachers to take advantage of the context to discuss common misconceptions about doing mathematics. With today’s emphasis on standardized testing and individual grades, it is easy for students to misunderstand the collaborative nature of mathematics. Group work offers students an opportunity to experience the social nature of mathematics. The increasing complexity of the problems allows students to experiment with the often-circuitous nature of authentic mathematics (Schoenfeld, 1992). We have tried to structure the problems so as to start students with written work including appropriate amounts of extraneous information so as to not overload cognitive load while simultaneously using problems that offer significant challenge, and that introduce students to the purpose and
utility of the mathematics they are learning (Ainley et al., 2006; Beswick, 2011; Cobb, 2007).

As with some of the other lessons in this thesis, the author advocates written work requiring students to explain their thinking which can invite inspection of the problem solving process (Connolly & Vilardi, 1989; Countryman, 1992; Maimon & al., 1989; Pugalee, 2001). This lesson expands the use of group work to require students to articulate their process, ask for feedback, and critique each others’ work, further eliciting metacognitive processing (Goos et al., 2002).
Lesson 6: Pre-Calculus

Big Idea: Graphing a function and its inverse.

Focus Question: What relationship do graphs of inverses appear to have? Do they look parallel, perpendicular, similar in shape, or have another relationship? Are they a rigid transformation?

NYS Common Core Standards:
G-CO.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

Objectives:
Students will know: the relationship between the graph of a function and its inverse.
Students will be able to: graph a function and its inverse.

Assessment/Evidence:
Performance Tasks: Students graph functions and their inverses.
Other Evidence: Students discuss and write about their observations.

Mathematical Practice Standards:
MP1. Make sense of problems and persevere in solving them.
MP3. Construct viable arguments and critique the reasoning of others.
MP5. Use appropriate tools strategically.
MP6. Attend to precision.
MP7. Look for and make use of structure.
MP8. Look for and express regularity in repeated reasoning.

This lesson is designed to deepen student understanding of inverses while building on their understanding of the role that slope plays in parallel and perpendicular lines. We view inverses through the lens of rigid transformations. Students work in groups with tracing or transfer paper to manually transform functions and analyze their results drawing conclusions and making connections in the process. We assume that students have recently covered horizontal and vertical shifts, and reflections of conic sections. Through this activity, students tie together several slope-related concepts from multiple courses: slope itself from 8th grade, parallel and perpendicular lines from
Geometry, transformations introduced in Algebra, and they receive a hint of limits and
derivatives in Calculus.

The material covered in this lesson can be challenging for students.
Transformations combined with inverses present a heightened demand on student
cognitive load. That said, research supports introducing students to challenging problems
(Ainley et al., 2006; Beswick, 2011; Cobb, 2007). By inviting students to apply
fundamental concepts such as inverses in a new way, we can shore up their foundation
and extend their capability to reason. We address the high cognitive load by utilizing
transfer paper to visualize the change that occurs as we apply transformations to
functions. By offering students a tactile and concrete method for producing graphs of
functions, we are scaffolding their problem solving process. Instead of visualizing the
transformations completely in their minds, the transfer paper enables them to simplify
this process and focus on the concept of inverses. Once their understanding of inverses-as-transformations solidifies, they are better prepared to graph functions and their
inverses without transfer paper. This is an excellent context for teachers to reinforce
cultural elements of doing real mathematics. Remind students that real mathematics is
challenging and time consuming, and requires persistence. Collaboration and multiple
varied attempts aid success. Problems do not always have “one right answer” and
mathematics can often be done in multiple ways as evidenced by the transformation
approach to inverses. By adopting these attitudes, students are increasingly aware of their
problem solving process and more likely to succeed (Schoenfeld, 1992).

This lesson is presented as an inquiry worksheet (contained in the Appendix) for
students to do in small groups. The beginning examples invite students to examine linear
functions to determine the equation of the inverse, and then graph both the original linear
function and the inverse. Opportunities are provided for students to discuss and write about their observations. Students make connections and draw conclusions about the properties of the graph of a function and its inverse with special attention paid to slope. Teachers may wish to stop the class after questions 5, 7, and 8 to check for understanding as these are pivotal questions. Toward the end of the activity, we hint at the limiting process and the slope of a tangent line that students will see in Calculus. By providing students with a hint of what is to come, we introduce students to the purpose and utility of the mathematics they are learning which in turn aids metacognition (Ainley et al., 2006; Beswick, 2011; Cobb, 2007). We close with analysis of worked problems requiring articulation of evidence of mathematical thinking (van Merrienböer & Sweller, 2005). Students see a transformation done correctly and are asked to articulate the process involved. Students then see an example done incorrectly and students are asked to determine what was done. These exercises can help students be more aware of their own process as well as their errors. We encourage frequent error analysis to acclimate students to the circuitous process of trial and error one often finds in higher mathematics (Schoenfeld, 1992). A familiarity with error analysis can prove to be most helpful as even the best of students make mistakes. An ability to find mistakes and self-regulate thinking throughout the correction process can serve students well.

We will assume that students have some familiarity with using transfer paper for transformations in previous lessons on reflections and horizontal and vertical shifts. Additionally, at this level, we assume that students have extensive experience sketching their own axes and graphs on graph paper and have good working familiarity with the definition of a function. We recommend that students use their own graph paper (or classroom supplies). The worksheet in the appendix includes the exercises but not graphs.
as on previous lessons. As with all lessons, a key is provided. This function touches on inverses that need their domain restricted. If this has not been covered yet, then that would make an excellent companion or follow-up topic. Additionally, piecewise functions would easily follow this content. Please note that many schools cover reflections in the line $y = x$ thoroughly in previous years. If students are well prepared in this technique, the teacher may adapt the lesson and emphasize at the beginning of the inquiry worksheet that we are searching for another transformation that accomplishes the same result. Additionally, teachers may allow students to divide up the questions to quicken the pace of work and allow students to speed through material that would be review and then focus on the more important and newer points.
Lesson 7: Calculus

Topic: Definition of the Derivative

Focus Question: How can we find the slope of a curve at a point on the curve?

Big Idea: Limit Definition of a Derivative

NYS Common Core Standards:
MP1. Make sense of problems and persevere in solving them.
MP2. Reason abstractly and quantitatively.
MP3. Construct viable arguments and critique the reasoning of others.
MP4. Model with mathematics.

AP Calculus Standards:
EU 2.1: The derivative of a function is defined as the limit of a difference quotient and can be determined using a variety of strategies.
LO 2.1A: Identify the derivative of a function as the limit of a difference quotient.
LO 2.1C: Calculate derivatives.
EK 2.1Aa: The difference quotients $\frac{f(a + h) - f(a)}{h}$ and $\frac{f(x) - f(a)}{x - a}$ express the average rate of change of a function over an interval.
EK 2.1A2: The instantaneous rate of change of a function at a point can be expressed by $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$ or $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, provided that the limit exists. These are common forms of the definition of the derivative and are denoted $f'(a)$.
EK 2.1A3: The derivative of $f$ is the function whose value at $x$ is $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$ provided this limit exists.

Objectives:
Students will be able to: find the slope of a line tangent to a curve at a point given a graph.
Students will be able to: find the derivative of a function using the limit definition of a derivative.
Students will be able to: find the equation of a tangent line to a curve at a given point.

This lesson draws on years of experience with the concept and strand of slope.

Students are introduced to the secant line to a curve, determine its slope, and begin to work with a formal definition of the derivative. New York State has no content standards for Calculus; however, the Mathematical Practice Standards most definitely apply in this context. Standards for AP Calculus from the College Board are cited above (College
Board, 2017). We assume that by this point, in their mathematics education, students will have encountered many of the teaching methods advocated above. Students will hopefully be prepared to do mathematics in an environment of discussion, deep questioning, and collaboration, which resembles how mathematics is done at the college level and beyond. The author has made the assumption that students at this level have developed clear note taking skills and are self-directed and motivated learners. As a result, no worksheets are provided in the Appendix to accompany the AP Calculus lesson. Teachers may wish to provide students with a copy of the image accompanying the ski example below, however beyond that, students hopefully will be able to take notes documenting the development of ideas throughout the lesson and discussion outlined below. Teachers may, of course, adapt the lesson to suit the particular group of students in their classes and provide more support.

As a warm up activity, students can work in small groups or with a partner to respond to the prompt: “How would you define a line tangent to a curve? Create your own definition of a tangent line.” We recommend that the teacher have several images on hand either on a handout or for reference on a SMART Board for discussion. The Sample Graphs in the Grade 8 lesson could serve as entertaining and instructive images. We next recommend that the class attempt to create an equation for a tangent line given a point on a curve, \((x_1, y_1)\). This should be done easily using the point-slope formula:

\[
(y - y_1) = m(x - x_1).
\]

In fact, in general, the equation of the tangent line will be exactly the point-slope formula. It can be helpful to remind students at this point that the point-slope form represents exactly the same relationship as the slope formula. By dividing both sides of the point-slope form by the quantity \((y - y_1)\), we can obtain the formula for
finding slope. Remember that the formula for finding the slope between two lines is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$  

Discussion can then lead toward approximating the slope at a point using a secant line and the limiting process that makes that secant line better approximate the curve. Lead students to develop the role of taking a limit through discussion and questions such as: “When will a secant line give us an accurate approximation of the curve? What will make the secant line a better fit? What processes could this line better reflect the true slope of the curve at a given point?” Note that by using the secant line, the given function facilitates calculations. The teacher should clarify this point and ensure that students understand how a secant line allows us to approximate the tangent line. We can define the tangent line as the limit of the secant line as the quantity \((x - x_1)\) goes to zero, or as the line passing through the given point with a slope equal to the limit of the slope of the corresponding secant lines.

Next, the class should (and we recommend this as a whole-class activity) consider the secant line connecting the points \((c, f(c))\) and \((c + \Delta x, f(c + \Delta x))\). What is the point-slope equation of this line? \(y - f(c) = m(x - c)\) What is the slope of this line? We find that \(m = \frac{f(c + \Delta x) - f(c)}{c + \Delta x - c}\). These are important exercises to work through together as they will support students as they work to build the derivative construct and make connections between point-slope form, the formula for the slope between two points, and the definition of a derivative.

We then recommend the traditional route of using the slope formula to find the slope between two points, \((c, f(c))\) and \((c + \Delta x, f(c + \Delta x))\). We found
that \( m = \frac{f(c + \Delta x) - f(c)}{c + \Delta x - c} \) which reduces to \( m = \frac{f(c + \Delta x) - f(c)}{\Delta x} \). By taking the limit of this expression as \( \Delta x \) goes to zero, we are able to define the derivative. We also can formally define the difference quotient equivalent to the change in \( y \) divided by the change in \( x \), and formally define the tangent line, the line passing through the given point with a slope equal to the difference quotient at that point. Teachers may wish to align these formal definitions with the text used in class.

We next propose an authentic context problem. A winter resort management company has just purchased a ski resort to add to their collection of properties. There is an old ski jump on the property. The new management wants to know if they can easily adapt the jump to competition standards and add competitions to the new attractions at their new resort. Competition standards require that the slope of the jump at the point of departure be between 0.07778 and 0.13333. Examine the image and data below. Using the slope of a secant line near the tangent line, make a judgment call. Do you have enough information to determine whether this ski jump can potentially be used for competitions? Why or why not? If not, what other information would you need?

(Wikipedia, 2017)
A surveyor has measured the elevation at the left hand end of the green “jump” section of the slope to be 4,326 feet. The elevation at the actual edge of the green section is 4,319.4 feet. The horizontal length of the green section is 32.7 feet. Students should discuss this in small groups preparing to present their conclusions to the class and defend their reasoning.

The slope of the secant line connecting these two “points” is -0.2018 if we were to superimpose a coordinate plane on the figure above. This figure presents several opportunities for discussion and attention to detail. Did students find a negative or positive slope? What does the sign mean in this context? The calculated slope of the secant line is too steep for Olympic standards. Do the surveyed points offer an accurate measurement of the slope of the ski jump at the point of departure? Why or why not? As students are at a more mature and independent level, we will leave it to each teacher to determine whether written structure in the form of a worksheet or formal notes are required. Students should be able to conclude that measurements closer to the point of departure will yield a more accurate slope. The secant line will be less steep and more likely to be Olympic standard if the measurements are closer together. A more precise survey could confirm that the slope is up to Olympic standards. Essentially we are using a real life situation to invite students to think about how a secant line can approximate the slope of a curve, and finding the limit as the secant line more closely approximates the tangent line will find a more accurate value for slope at the point of tangency. We recommend that the teacher circulate the classroom listening and asking probing questions to stimulate discussion and elicit metacognition and articulation of the above concepts. Schoenfeld’s questions could be particularly beneficial in this context: “What (exactly) are you doing? (Can you describe it precisely?) Why are you doing it? (How
does it fit into the solution?) How does it help you? (What will you do with the outcome when you obtain it?)” (1992, p. 63). By asking questions and requesting clarifying explanations, the teacher is modeling metacognitive activity for students.

We now shift back to theory and practice finding the slope of a tangent line using specific function examples. First the teacher can model the process using a simple linear function such as $f(x) = 3x - 2$. Before calculating the slope of the tangent line using the definition of the derivative, ask students what they think the slope of the tangent line will be? This should be a simple but hopefully profound “aha” moment as they realize that the tangent line to a linear function will be the linear function itself. Next consider a quadratic function again first asking students to predict or make a guess at the slope of the tangent line: $f(x) = 3x^2 + 1$. The teacher should model substituting the function into the limit definition of the derivative, asking questions along the way at each step to invite students to offer potential steps. The teacher should also model metacognitive strategies by thinking out loud. Discuss potential next steps before committing to a course. Discuss the pros and cons of various options. There should be a running dialogue among students and teacher about the process. Once the derivative is found to be $6x$, the teacher can ask the students to discuss what this may mean. What does it mean when the slope depends on the value of $x$? What if $x = 2$ or $0$ or $-2$? What would the tangent line look like? Invite students to draw on a graph of the equation on a SMART Board or other large visual representation. Watch for opportunities to discuss and correct errors making sure to validate course corrections as legitimate and normal processes in mathematics.

As appropriate in the context of the individual teacher’s objectives, teachers may also formally define and discuss the concept of differentiability and its relationship to the derivative. This important concept should be covered in class, but the teacher’s individual
judgment can determine whether it is done on the same day or saved for the next.

Teachers also will want to introduce the exceptional case of the slope of vertical tangent line. This can be done in the context of an example the same day or the next, or an anticipatory warm up exercise or homework question can lead into the topic to be covered the next day.

Further examples with polynomials such as \( f(x) = x^3 - 2x \) are recommended for practice using the limit definition of a derivative. The same functions can then be used to find the slope of a tangent line at a given point. This is best done using the point-slope form as the given point will rarely be a y-intercept. Inviting students to propose a method for finding the slope of a curve as well as an equation for a tangent line at a given point can follow these examples. For example, find \( f'(3) \) using \( f'(x) \) if \( f(x) = x^3 - 2x \). Two challenge problems could include finding the derivative of \( f(x) = \sqrt{x} \) and \( g(x) = \frac{1}{x} \).

Extension topics for the same day could include the role of continuity in the derivative, as well as the need for one-sided limits to exist on the left and on the right. Alternatively, these topics could be addressed in the following days. Students should have the opportunity to consider piecewise functions such as \( f(x) = 2 \) if \( x < 0 \); \( f(x) = 4 \) if \( x \geq 0 \). This would be a discussion-rich example for group work. With students in groups of 4 or 5, students can consider the following: Is the function differentiable at \( f(0) \)? The teacher can draw a student’s name from a bag or box. By introducing an element of chance, the teacher will ensure that students all talk together and work to understand the application of the derivative to piecewise functions. Teachers may also wish to consider a graph with a sharp turn or a point such as \( f(x) = |x| \). At this point, we propose several summary questions for the classroom teacher to choose from for use as an Exit Ticket: Explain how
to find the slope of a curve at a point and how this process is different than finding the slope of a straight line. Explain the relationship between secant and tangent lines, and the derivative. Explain why we must take a limit as one step in finding the derivative of a function. Explain why a function must be continuous at \( f(x) \) in order to be differentiable at \( f(x) \). Give an example of a function that is continuous but not differentiable at a point. Sketch the curve of the function and explain why it is not differentiable. By articulating their reasoning in writing, the teacher invites metacognition and allows students to solidify their own thinking and learning (Schoenfeld, 1992).

We conclude by pointing out that the power of this lesson depends on the success of all of the above lessons. The derivative concept is the culmination and universal application of the unit rate and slope concept. Without a firm understanding of both middle school concepts, thorough comprehension of the derivative and its powerful applications is impossible. Furthermore, the derivation, calculation, and application of the derivative depend on student comprehension of the point-slope formula (Algebra 1) of a line as well as strong graphing skills related to slope (Geometry and Precalculus). In order to understand the extensive potential in real-life situations, a solid grasp of linear and other regression processes is extremely helpful (Algebra 1I). The concept of unit rate permeates all of the above lessons and ties them together leading to and culminating in the derivative.
AP Statistics

The Advanced Placement Statistics course lies well outside the CCLS so we determined that it was not appropriate to provide a lesson in this context. No lesson development or worksheets are provided on this topic. It is a significantly different discipline than Calculus with different notation and different goals. Many of the above lessons and concepts do, however, lead naturally to greater success in an advanced statistics course such as AP Statistics. We recommend that secondary teachers and Algebra 2 teachers in particular look ahead to AP Statistics curricula with a goal to be informed about what students need to know in order to be successful. Many topics from AP Statistics that students struggle with have their foundation laid in Algebra 2. A strong understanding of slope, correlation, the correlation coefficient, and regression will better prepare students for challenging AP Statistics content. Furthermore, secondary instruction with a metacognitive framework can enable students to tackle complex problems with high cognitive load and different notation. AP Statistics involves the use of excessive amounts of extraneous information. Metacognitive strategies are well suited to this demanding challenge.
Validity

Three reviewers, all mathematics teachers in the same school district, read this thesis and offered feedback. Their feedback provided enormously helpful insights that reflected their expertise as classroom teachers. Reviewer A teaches 6th grade mathematics in a middle school setting and offered feedback on grades 7-8. Reviewer B teaches 8th grade in the same district and teaches 8th grade mathematics. She teaches both the students following the standard curriculum as well as the accelerated students taking Regents Algebra 1 and was able to offer feedback on her areas of expertise as well as general feedback on the entire thesis. Reviewer C is the current high school mathematics department chair, has extensive experience with high school mathematics courses, and currently teaches honors precalculus and AP Calculus (BC). She offered feedback on all lessons as well.

All three reviewers offered helpful feedback on wording or typos and for the sake of brevity and focus, those suggestions were incorporated into the final product but will not be outlined here. All reviewers also noted that the timing of various activities may need to be adjusted depending on various classroom needs. The author encourages teachers to use their classroom experience and knowledge of their students’ individual needs to adjust and adapt activities to fit the time available in their own classrooms. Some lessons can be easily spread out over more than one day if teachers feel that better meets their needs.

Reviewer A felt the thesis topic was very timely and aligned well with New York State standards and Common Core standards as she sees them applied and used in her school. The teaching methods presented and the focus on metacognition fit well with research-supported methods that she has learned in other settings. She appreciated the
reminder of metacognitive strategies she had previously seen, as well as new ideas, as
these often get forgotten in day-to-day struggle to get assignments prepared and stay on
top of the day to day tasks of running a classroom. Reviewer A felt that the practice work
on positives and negatives included in the 8th grade lesson was appropriate and important.
She definitely sees a need for teachers to offer extra support in this problem area as she
sees students struggle frequently with signs in arithmetic. The initial draft submitted for
review had questions with tables and questions with graphs, but no questions that
synthesized both elements in the same question. The worksheet has been adjusted to
ensure that the last several questions require students to use tables and graphs together in
the same question enabling them to see how text, tables, and graphs can represent the
same information. Additionally, the commentary on “notes” versus “worksheets” at the
end of the 7th grade lesson came from Reviewer A’s insightful comments, as did the
warning about “slope” definitions and the PBS video used in the 8th grade lesson.

Reviewer A also was able to explain and clarify a progression of slope related
concepts from 6th through 8th grade. In her experience, students learn about unit rate in 6th
grade, and then build on that understanding in 7th grade as they learn about the constant
of proportionality. Reviewer B will in fact sometimes use the language “constant of
proportionality” in her 6th grade classrooms and occasionally the word “slope” to expose
her students to that particular language to prepare them for future concepts. Middle
school lessons in 7th and 8th grade then build on a basic understanding of unit rate,
progressing to the constant of proportionality and slope, culminating with a solid
development of the concept and a formal definition of slope in the 8th grade. This
discussion and personal experience from Reviewer B informed language in both the
above text and the worksheets that refers to unit rate, slope, and especially the constant of proportionality.

Overall, Reviewer A felt that the lessons offered a strong cohesive vision of the concept of slope for educators built on appropriate and proven methods. Just this summer as a parent, she watched her recently graduated daughter practicing for her college mathematics placement exam. Her daughter made some of the very same careless mistakes addressed above and Reviewer A wondered if her own daughter could benefit from greater focus on metacognition to avoid errors made by misapplying frequently used algorithms.

Reviewer B offered several points of positive feedback: She noted appropriate “anticipation of where students may struggle (positive and negative signs for example) and how to address this.” She appreciated permission for students to make mistakes and noted that it is an important “part of the learning process.” She felt the lessons and group work activities were well planned and appropriate for the topics and grade levels. She particularly appreciated the use of flipped lessons in the later grades. She and her fellow mathematics teachers have discussed flipped lessons several times at math department meetings. In general, they do not feel that flipped lessons would work well for all lessons at the middle school level (she noted that the use of flipped instruction at the high school level as presented in this thesis was appropriate). She would, however, like to incorporate a few flipped lessons in her next year of teaching since all students in the building will have their own laptops starting this fall. Reviewer B offered helpful context that clarified for the author the connection between “slope” and “rate of change.” This conversation informed the language in the 8th grade lesson on slope as a rate of change. Additionally, Reviewer B informed me that her school uses triangles to demonstrate that
the slope of a line is constant between any two points on a line. This technique provides a method for students to provide their own scaffolding and support slope calculations from a graph. The worksheets have been adjusted to incorporate this very helpful strategy.

Overall, Reviewer B felt the lessons did “a great job connecting the concept of slope/rate of change throughout all grade levels, taking into account previous knowledge” and that this thesis demonstrates “a great understanding of the learning process for students and how teachers can help their students to achieve success.”

Reviewer C offered thorough and detailed feedback including helpful formatting and wording suggestions. She suggested points in lessons to include follow up questions and further discussion with students and these suggestions have been incorporated into the lessons. Reviewer C commented as did Reviewer A on clarifying the definition of slope as used in the PBS video referenced in the 8th grade lesson. She also noted the appropriate and much needed practice with positive and negative numbers in the same lesson. In the Algebra 2 lesson, she pointed out that some schools use ax + b for regression instead of a + bx which was in the thesis draft submitted for review. This has been noted in the text. Additionally, draft worksheets for the Algebra 2 lesson discussed on the value of r but not the meaning of the sign of r. This has been corrected on the worksheets. Her suggestion added to the text in the precalculus lesson to check for understanding after questions 5, 7, and 8 will help teachers keep students on track and aid understanding. Reviewer C also noted that the work for synthetic division on question 13 was omitted and this has been rectified. Her final note on the precalculus lesson voiced concerns that students may be bored with parts of the lesson since they have been exposed to reflection over the line y=x in previous courses. Suggestions to avoid this
have been added to the above text. In the draft lessons submitted for review, the calculus lesson referenced only the Common Core Mathematical Practice Standards. Reviewer C kindly drew the author’s attention to existing standards produced by the College Board for AP Calculus courses. She particularly enjoyed the development of the difference quotient formula and the leading questions posed to students throughout the lesson. She felt the ski question was at an appropriate level for the course and the students. She also offered helpful suggestions on deepening the discussion leading to the development of a tangent line from the limit of a secant line as $(x - x_i)$ and suggestions clarifying the justification for not including worksheets beyond precalculus. Reviewer C did not feel confident commenting on the AP Statistics section above, which reinforced the author’s decision to consider statistics to be a separate domain and to omit lesson materials on that topic.
Conclusion

It is this author’s hope that this curriculum will prove to be a helpful and productive resource for teachers as they develop their students’ problem solving skills. This curriculum can serve as a resource for teachers in need of CCLS-aligned materials. It offers a solid foundation for building a strong understanding of slope. The NYS standards highlight the interrelatedness of middle school and high school mathematics:

The evidence concerning college and career readiness shows clearly that the knowledge, skills, and practices important for readiness include a great deal of mathematics prior to the [topics explicitly outlined] in these standards. Indeed, some of the highest priority content for college and career readiness comes from Grades 6-8. This body of material includes powerfully useful proficiencies such as applying ratio reasoning in real-world and mathematical problems, computing fluently with positive and negative fractions and decimals, and solving real-world and mathematical problems involving angle measure, area, surface area, and volume (NYSED, 2017, p. 72).

We propose that the foundational applications of ratio to unit rate and slope in the middle school years prove to be pivotal in looking ahead to greater mathematical success. Furthermore, the metacognitive strategies offered can be beneficial throughout students’ remaining mathematics classes both as they apply to slope as outlined above and applied to other mathematical topics throughout secondary courses and beyond. It is the author’s hope that the curriculum examples using metacognitive strategies will serve as a guide for increasing student awareness of mathematical processes and that achievement and motivation will follow, and that teachers using this curriculum will get closer to that elusive goal of every student reaching success.
References


https://en.wikipedia.org/wiki/Ski_jumping_hill


Appendix A

Common Core Standards of Mathematical Practice As Utilized by the NYS

Department of Education

1. Make sense of problems and persevere in solving them.
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. Reason abstractly and quantitatively.
Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3. Construct viable arguments and critique the reasoning of others.
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. Model with mathematics.
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships
mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. Use appropriate tools strategically.
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. Attend to precision.
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. Look for and make use of structure.
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see $7 \times 8$ equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as $2 \times 7$ and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers $x$ and $y$.

8. Look for and express regularity in repeated reasoning.
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1, 2)$ with slope $\frac{y}{x}$ and $(x-1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x-1)(x+1)(x-1)(x^2+x+1),$ and $(x-1)(x^2+2x+1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content
The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.
The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics (NYSED, 2012).
Appendix B

Lesson Materials

Grade 7 Worksheet

Name ______________________________

1. If I buy a bag of miniature candy bars to use as prizes in our class, and the bag contains, 300 pieces, how many pieces (on average) could ONE person receive?

2. An 8-ounce can of peaches costs $0.79 and a 12-ounce can of peaches costs $1.09. Calculate the unit price per ounce to tell which is the better deal.

3. The wind ensemble wants to purchase t-shirts for their next band trip. Using the table below, find the unit price for ONE t-shirt.

<table>
<thead>
<tr>
<th>Number of t-shirts purchased</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$42.50</td>
</tr>
<tr>
<td>10</td>
<td>$85.00</td>
</tr>
<tr>
<td>15</td>
<td>$127.50</td>
</tr>
<tr>
<td>20</td>
<td>$170.00</td>
</tr>
</tbody>
</table>
4. The table below represents the average number of candy bars needed depending on how many people come trick-or-treating at my door. How many candy bars am I giving to each person?

<table>
<thead>
<tr>
<th>Number of people</th>
<th>Number of Candy Bars</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>20</td>
<td>60</td>
</tr>
<tr>
<td>30</td>
<td>90</td>
</tr>
<tr>
<td>40</td>
<td>120</td>
</tr>
</tbody>
</table>

5. The graph to the right represents how many loads of laundry Mom needs to do depending on the number of children home (or visiting) from college during spring break. How many loads of laundry does Mom do for each additional child? Create a table to aid your calculations.

<table>
<thead>
<tr>
<th>Number of Children</th>
<th>Laundry Loads</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
6. The graph to the right shows how much pizza Fred needs to order depending on how many people will be at his party. Create a table reflecting the data on the graph. Find out how much pizza he is planning on each person eating.

<table>
<thead>
<tr>
<th>Number of People</th>
<th>Pizzas</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. Cosgrove Middle School has planned a class trip to Boston. 352 students are going on the trip. School district policy requires one chaperone for every 8 students. Create a table of values and a graph of this relationship. Be sure to label your table and graph clearly. How many chaperones will be needed?

<table>
<thead>
<tr>
<th>Number of People</th>
<th>Chaperones</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
8. The graph to the right represents the distance a family rides on their bike ride along the canal. Create your own table representing the graph. How many miles do they typically drive in one hour? Find \( r \) for the point \((1, r)\). What does this point represent?

Define Unit Rate: ____________________________________________________________________________

Give three examples of a unit rate: ____________________________________________________________

Why did you choose these examples? ____________________________________________________________________________

<table>
<thead>
<tr>
<th>Number of hours</th>
<th>Miles ridden</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
</tr>
</tbody>
</table>
Grade 8 Inquiry Activity

1. At what speed (in miles per hour) was the car traveling if the table below represents distance traveled and time for Juanita on her road trip to Albany?

Remember that \( v = \frac{d}{t} \).

<table>
<thead>
<tr>
<th>Distance (miles)</th>
<th>Time (minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>23</td>
</tr>
<tr>
<td>55</td>
<td>51</td>
</tr>
<tr>
<td>115</td>
<td>106</td>
</tr>
</tbody>
</table>

a. Plot the data above on the graph below. (Remember that you cannot control Time so it should be placed on the x-axis as your independent variable.) Don’t forget to label your axes!
b. What is Juanita’s speed during the first 23 minutes of her drive in miles per minute? Draw a triangle on your graph representing this change.

c. What is Juanita’s speed during the first 51 minutes of her drive in miles per minute? Draw a triangle on your graph representing this change.

d. What is Juanita’s speed during the first 106 minutes of her drive in miles per minute? Draw a triangle on your graph representing this change.

e. Did you get the same answers for a, b, and c? Explain.

f. Approximately how many miles can Juanita drive in 1 minute? Round to the nearest mile.

g. Approximately how many miles can Juanita drive in one hour? Round to the nearest mile.
2. The graph below represents how many pages Fred has typed at various points during the day while working on a paper for school. Approximately how many pages is Fred typing every 5 minutes? Draw a triangle on your graph representing this change. What is Fred’s unit rate expressed in pages per minute? Explain any connections you see between the unit rate and the graph below.
3. This graph represents the number of cookies eaten \((y)\) and the number of people at the table \((x)\). Use this graph to answer all of the questions below.

![Graph showing the relationship between the number of cookies eaten and the number of people at the table.]

a. Find the unit rate. Draw a triangle on your graph to help you with your calculations. What does the unit rate represent?

b. What does the point \((0, 0)\) represent? What do the points \((1, 3)\) and \((2, 6)\) represent?

c. We can write some of the above values as ratios. 3 cookies to 1 person, or 6 cookies to 2 people. How else can we represent these ratios?
d. Suppose there were 12 people eating a lot of cookies. How many cookies would you expect? How would you write these numbers in a ratio similar to the above ratios? Simplify your ratios.

e. How do the above ratios compare to each other? What happens to this ratio as we look at different sized groups of people?

Slope: _________________________________________________________________

_______________________________________________________________________

Slope can be found using this formula:
Sample Graphs

These graphs can be used for a brief discussion on the power of slope at a point in real life situations. Teachers can choose examples that best fit the interest of their students.

Example 1: This diagram from the roofing industry is technically above the ability of 8th graders however it quickly illustrates the real-life application of the concept of slope in a hands-on career not requiring a college education such as construction, or a more technically advanced career such as engineering or architecture.

(Williams, 2017)

Example 2: We recommend sketching a quick tangent line on the image while entertaining the questions “How can we find the slope of this curve?” and “What would slope represent?”
Example 3: This graph may be compelling for students interested in action and adventure sports.

(Mallon, 2014)

It represents official Olympic ski jumping standards as dictated by the International Ski Federation. A quick tangent line drawn on a SMART Board can aid discussion about how slope affects speed, and how mathematics plays an important and exciting role in sports.
Example 4: This graph may be particularly energizing for students who appreciate gross-out humor. It represents the population growth of lice on sheep raised for wool production.

(Government of Western Australia Department of Agriculture and Food, 2017)

The sheep caretaker measures the lice population by combing a 10 centimeter part in the sheep’s wool and then counting the number of lice visible in that part. Caretakers must then weigh the cost of treating the sheep and cleaning the wool against the cost of disposing of infected wool and lost income. A lively class discussion can ensue about the need to catch the infestation before the slope of the line increases past the mostly flat initial slope stage.
Grade 8 Error Analysis

Name ____________________________

Find the mistake(s) in the work below. Correct the mistake(s) and explain the error or errors.

1. Find the slope of the line connecting the points (-2, -7) and (3, 5).

   \[ m = \frac{5 - (-7)}{3 - (-2)} = \frac{12}{5} \]

   \[ m = \frac{-2}{1} \]

   \[ m = -2 \]

2. Find the slope of the line connecting the points (4, 2) and (2, -4).

   \[ m = \frac{4 - 2}{2 - 4} = \frac{2}{-2} \]

   \[ m = \frac{-6}{2} \]

   \[ m = -3 \]
**Algebra 1 In-class worksheet:**

Name: ________________________

**K-W-L Chart**

Writing an equation for a straight line:

<table>
<thead>
<tr>
<th>Know—what do you already know about finding the equation of a straight line?</th>
<th>Wonder—what do you wonder about? What have we not yet figured out how to do? When can we NOT yet find the equation of a line?</th>
<th>Learn—what have you learned about writing equations for lines?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Geometry Inquiry Worksheet

1) Graph \( y = 3x + 1 \) and \( y = 3x - 2 \) below:

2) Graph \( y = -x - 1 \) and \( y = -x - 4 \) below:

3) Graph \( y = -\frac{1}{2}x - 3 \) and \( y = -\frac{1}{2}x + 4 \) below:

4) Graph \( y = \frac{1}{2}x + 3 \) and \( y = \frac{1}{2}x - 1 \) below:
5) Summarize any patterns you see above: _____________________________________________
_______________________________________________________________________________
________________________________________________________________________________
6) Discuss this pattern with your group and come to a consensus. Describe your group conclusions:
________________________________________________________________________________
________________________________________________________________________________

Before continuing, call your teacher over to discuss your results.

7) Graph $y = 3x + 1$ and $y = -3x - 2$ below:  

8) Graph $y = -x - 1$ and $y = x - 4$ below:
9) Graph $y = -\frac{1}{2}x - 3$ and $y = \frac{1}{2}x + 1$ below:  

10) Graph $y = \frac{1}{2}x + 3$ and $y = -\frac{1}{2}x - 1$ below:

11) Summarize any patterns you see above: ____________________________________________
________________________________________________________________________________
________________________________________________________________________________

12) Discuss this pattern with your group and come to a consensus. Describe your group conclusions: _____________________________________________________________________
________________________________________________________________________________
________________________________________________________________________________

Before continuing, call your teacher over to discuss your results.
13) You are given the line $y = 3x + 4$. Graph the line. Then draw another line parallel to this line. What is the equation of this second line?

14) You are given the line $y = -\frac{1}{2}x + 1$. Graph the line. Then draw another line perpendicular to this line. What is the equation of this second line?

Summary:

Parallel lines have slopes that are _________________.

Perpendicular lines have slopes that are _________________.

Exit Ticket

Name ______________________________________

Explain how can you tell if two lines are parallel: _______________________________
_______________________________________________________________________

How can you tell if two lines are perpendicular: _________________________________
_______________________________________________________________________

Parallel or Perpendicular? Y = -3x + 5 and y = 1/3x -5 __________________________

Parallel or Perpendicular? Y = 2x – 2 and y = -1/2x + 2 ___________________________
Algebra 2 Inquiry Worksheet

All questions should be worked in your group. Your goal is discussion that results in consensus. Remember that your calculator output for linear regression gives you “a” and “b” where a represents your slope and b represents your y-intercept for a line in the form \( y = ax + b \), which is an alternate form of a slope-intercept equation.

1. The table below presents population census data for the city of Rochester (Census Bureau, 2017).

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980</td>
<td>241,741</td>
</tr>
<tr>
<td>1990</td>
<td>231,636</td>
</tr>
<tr>
<td>2000</td>
<td>219,773</td>
</tr>
<tr>
<td>2010</td>
<td>210,585</td>
</tr>
</tbody>
</table>

a. Using the given data, find the equation of the linear regression line. Using full sentences, explain the meaning of the output values from your calculator. Remember \( r \) tells you how closely your equation fits the actual data as well as whether your data trends uphill (for positive \( r \) values) or downhill (for negative \( r \) values). The closer to 1 or -1, the better the fit.

Equation:

\[ a = \]

\[ b = \]

\[ r = \]
b. Using your equation, predict the population in the year 2015.

c. The official estimated population in 2015 was 209,802. How would you explain the discrepancy? Use full sentences. Would including this data change your regression equation? How?

d. Calculate a new regression equation using all of your census data and the 2015 estimated population. How did your equation change? Did it change in the way you expected it to? Explain in full sentences.
e. What is your new r-value? What does that tell you? Is it different than your old r-value? How? And what does that change mean?

f. What is the unit rate for this data set? Explain what this number means (using full sentences). Why would this be a useful number for city planners to know?

g. The US Government has census data for the city of Rochester going all the way back to 1820. Could including this information improve your prediction? Why or why not?
2. The table below presents the average height for children, both boys and girls, of various ages.

<table>
<thead>
<tr>
<th>Age</th>
<th>Average Height (boys)</th>
<th>Average Height (girls)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 years</td>
<td>31 inches</td>
<td>30 inches</td>
</tr>
<tr>
<td>3 years</td>
<td>33 inches</td>
<td>33 inches</td>
</tr>
<tr>
<td>4 years</td>
<td>37 inches</td>
<td>37 inches</td>
</tr>
<tr>
<td>5 years</td>
<td>40 inches</td>
<td>40 inches</td>
</tr>
<tr>
<td>6 years</td>
<td>42 inches</td>
<td>41 inches</td>
</tr>
<tr>
<td>7 years</td>
<td>44 inches</td>
<td>43 inches</td>
</tr>
<tr>
<td>8 years</td>
<td>45 inches</td>
<td>45 inches</td>
</tr>
<tr>
<td>9 years</td>
<td>49 inches</td>
<td>47 inches</td>
</tr>
<tr>
<td>10 years</td>
<td>51 inches</td>
<td>51 inches</td>
</tr>
<tr>
<td>11 years</td>
<td>52 inches</td>
<td>52 inches</td>
</tr>
</tbody>
</table>

a. Find the linear regression equation for the above data. You choose boys or girls.

b. What does slope represent in your equation? What does the y-intercept represent? Do you think these are accurate figures?
c. Compare your results with other groups. For groups that used the same gender, did they get the same equation? If not, troubleshoot. For groups that found a regression line for the other gender, what equation did they get? How do the equations for both genders compare?

d. What is the unit rate for each data set? What does it represent? Explain in full sentences.

e. Use the above equations to estimate the height of an 18 year old of the both genders. Do you think these estimate reflect the reality you see around you at school? Why or why not?

f. What is your r-value? Interpret your r-value and how it could impact your answer to part e.
g. If not, what would help you calculate more accurate figures? What would happen to your r-value if your data were more accurate?

3. The data represented in the graph and table below represent the concentration of phenolphthalein as a chemical reaction progresses.

(Bodner, n.d.)

<table>
<thead>
<tr>
<th>Concentration of Phenolphthalein (M)</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0050</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0045</td>
<td>10.5</td>
</tr>
<tr>
<td>0.0040</td>
<td>22.3</td>
</tr>
<tr>
<td>0.0035</td>
<td>35.7</td>
</tr>
<tr>
<td>0.0030</td>
<td>51.1</td>
</tr>
<tr>
<td>0.0025</td>
<td>69.3</td>
</tr>
<tr>
<td>0.0020</td>
<td>91.6</td>
</tr>
<tr>
<td>0.0015</td>
<td>120.4</td>
</tr>
<tr>
<td>0.0010</td>
<td>160.9</td>
</tr>
<tr>
<td>0.00050</td>
<td>230.3</td>
</tr>
<tr>
<td>0.00025</td>
<td>299.6</td>
</tr>
<tr>
<td>0.00015</td>
<td>350.7</td>
</tr>
<tr>
<td>0.00010</td>
<td>391.2</td>
</tr>
</tbody>
</table>

a. Which data points in the table would enable you to calculate a helpful linear regression line to use to predict the concentration of phenolphthalein at 40 seconds? Justify your choice using complete sentences.
b. Find the equation of a linear regression point using the data you identified in part a.

c. Explain the meaning of your slope, your y-intercept, and the unit rate.

d. Graph your line on the graph above. Find another group and compare your line to another group’s graph. Which line would make more accurate predictions? Explain your answer.

e. List the r-values for both your equation and the other group’s equation below. According to r-values, which line is a better fit? Does this agree with your hypothesis from part d above? Why or why not?
f. The graph below adds a new line on top of our data. This line represents a line tangent to the data at a particular point. This line has the slope that the data would have at the point (51.1, 0.0030) if we zoomed in and were able to find the slope at that particular point. How would predictions using this line compare to predictions using your linear regression line? Discuss in your group. Explain your group’s conclusions below using complete sentences.
Exit Ticket

Name ________________________

Explain in full sentences the meaning of slope and y-intercept in the context of Linear Regression:

Explain in full sentences the role of Unit Rate in the context of Linear Regression:

Exit Ticket

Name ________________________

Explain in full sentences the meaning of slope and y-intercept in the context of Linear Regression:

Explain in full sentences the role of Unit Rate in the context of Linear Regression:
Precalculus Inquiry Worksheet

Remember cardinal rule of using transfer paper: **ALWAYS trace and label your axes before moving your transfer paper.** For each problem, first graph the function, and then graph its inverse. To graph the inverse, remember to first find the equation of the inverse by switching $x$ and $y$, solving for $y$, and then rewrite using function notation $f'(x)$. Your job will be to examine the relationship between the two and determine what type of transformations would lead you from the initial function to its inverse. You are also looking for a relationship between the slopes of the original and inverse functions.

1. $f(x) = x - 2$

   Inverse equation:

   Is the inverse a function? If not, how would you make it so?

   Is there a relationship between the slopes of the two sketches? Are they parallel, perpendicular, or neither?

2. $f(x) = 2$

   Inverse equation:

   Is the inverse a function? If not, how would you make it so?

   Is there a relationship between the slopes of the two sketches? Are they parallel, perpendicular, or neither?
3. \( f(x) = 3x + 2 \)

Inverse equation:

Is the inverse a function? If not, how would you make it so?

Is there a relationship between the slopes of the two sketches? Are they parallel, perpendicular, or neither?

4. \( f(x) = \frac{3}{2}x - 4 \)

Inverse equation:

Is the inverse a function? If not, how would you make it so?

Is there a relationship between the slopes of the two sketches? Are they parallel, perpendicular, or neither?

5. \( f(x) = -\frac{3}{5}x + 2 \)

Inverse equation:

Is the inverse a function? If not, how would you make it so?

Is there a relationship between the slopes of the two sketches? Are they parallel, perpendicular, or neither?

6. What relationship do you see between the graph of a function and its inverse?
7. Can you describe another way to transform the original function to the image represented by the inverse using transformations previously studied? Consider horizontal and vertical shifts, dilations, rotations, reflections, alone or in succession. (There may be more than one right answer.)

Graph the following functions the traditional way and then examine them with your transformation hypothesis from #5 above in mind.

8. \( f(x) = x^2 - 5 \)

Inverse equation:

Is the inverse a function? If not, how would you make it so?

Is there a relationship between the slopes of the two sketches? Are they parallel, perpendicular, or neither?

Choose one point on your function, and the inverse of that point. What if you zoomed in and approximated the curve for a small section or used a line tangent to a curve at that point?
9. \( f(x) = x^3 + x^2 - 16x - 16 \)

Inverse equation:

Is the inverse a function? If not, how would you make it so?

Is there a relationship between the slopes of the two sketches? Are they parallel, perpendicular, or neither?

Choose one point on your function, and the inverse of that point. What if you zoomed in and approximated the curve for a small section or used a line tangent to a curve at that point?

10. Does your transformation theory from #5 hold true? Do you need to make any adjustments? If you have not yet identified more than one way to use transformations to graph the inverse of a function, do so now.

11. Explain to another student in full sentences a shortcut for sketching the inverse of a function.

12. Summarize the relationships you observed between the slope (or approximate slope) of a curve and the slope of its inverse.
13. Error Analysis: A student has worked the following problems. Explain the student’s problem solving strategy. Determine whether the student has done the work correctly or incorrectly. If the student has made mistakes, identify and describe the mistakes, and then fix the mistakes. You may use transfer paper.

a. Graph the function and its inverse: \( f(x) = x^3 - 7x + 6 \)

\[
0 = x^3 - 7x + 6 \\
\text{Graph:}
\]

\[
0 = (x - 1)(x^2 + x - 6) \\
0 = (x - 1)(x + 3)(x - 2) \\
x = 1, -3, \text{ or } 2
\]
b. Graph the function and its inverse: \( f(x) = \ln(x + 2) \).

Since there are no zeros to solve for, the student in this example graphed the original function and its inverse using their knowledge of transformations. Examine the graph, and draw conclusions about the student’s thought process from the graph.
Appendix C

Keys

Grade 7 Worksheet

Name __________________________

1. If I buy a bag of miniature candy bars to use as prizes in our class, and the bag contains, 300 pieces, how many pieces (on average) could ONE person receive?

\[
\frac{300}{\text{# of students in class}}
\]

\[
\square \text{ pieces of candy per person}
\]

2. An 8-ounce can of peaches costs $0.79 and a 12-ounce can of peaches costs $1.09. Calculate the unit price per ounce to tell which is the better deal.

\[
\begin{align*}
\text{8 ounce can} & \quad \text{12 ounce can} \\
\frac{0.79}{8 \text{ oz}} & = 0.09875 \quad \frac{1.09}{12 \text{ oz}} = 0.090833 \\
& \quad \$0.10/\text{oz} \quad \$0.09/\text{oz}
\end{align*}
\]

The 12 oz can is a better deal

3. The wind ensemble wants to purchase t-shirts for their next band trip. Using the table below, find the unit price for ONE t-shirt.

<table>
<thead>
<tr>
<th>Number of t-shirts purchased</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$42.50</td>
</tr>
<tr>
<td>10</td>
<td>$85.00</td>
</tr>
<tr>
<td>15</td>
<td>$127.50</td>
</tr>
<tr>
<td>20</td>
<td>$170.00</td>
</tr>
</tbody>
</table>

\[
\frac{170.00}{20} = \$8.50/\text{t-shirt}
\]
4. The table below represents the average number of candy bars needed depending on how many people come trick-or-treating at my door. How many candy bars am I giving to each person?

<table>
<thead>
<tr>
<th>Number of people</th>
<th>Number of Candy Bars</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>30</td>
</tr>
<tr>
<td>20</td>
<td>60</td>
</tr>
<tr>
<td>30</td>
<td>90</td>
</tr>
<tr>
<td>40</td>
<td>120</td>
</tr>
</tbody>
</table>

5. The graph to the right represents how many loads of laundry Mom needs to do depending on the number of children home (or visiting) from college during spring break. How many loads of laundry does Mom do for each additional child? Create a table to aid your calculations.

<table>
<thead>
<tr>
<th>Number of Children</th>
<th>Laundry Loads</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
</tr>
</tbody>
</table>

Mom does 2 extra loads of laundry for each extra child in the house.
6. The graph to the right shows how much pizza Fred needs to order depending on how many people will be at his party. Create a table reflecting the data on the graph. Find out how much pizza he is planning on each person eating.

<table>
<thead>
<tr>
<th>Number of People</th>
<th>Pizzas</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
</tr>
</tbody>
</table>

He buys one pizza for every four people coming so Fred is planning on each person eating $\frac{1}{4}$ of a pizza.

7. Cosgrove Middle School has planned a class trip to Boston. 352 students are going on the trip. School district policy requires one chaperone for every 8 students. Create a table of values and a graph of this relationship. Be sure to label your table and graph clearly. How many chaperones will be needed?

<table>
<thead>
<tr>
<th>Students</th>
<th>Chaperones</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>100</td>
<td>12.5 (or 13)</td>
</tr>
<tr>
<td>200</td>
<td>25</td>
</tr>
<tr>
<td>352</td>
<td>44</td>
</tr>
</tbody>
</table>

They will need 44 chaperones.
8. The graph to the right represents the distance a family rides on their bike ride along the canal. Create your own table representing the graph. How many miles do they typically drive in one hour? Find \( r \) for the point \((1, r)\). What does this point represent?

\[
\begin{array}{c|c}
\text{hours} & \text{miles} \\
1 & 4 \\
2 & 8 \\
\end{array}
\]

\[ r = 4 \text{ miles/hour} \]

\( r \) is the number of miles they drive each hour or their speed.

Define Unit Rate: A comparison of two numbers in ratio form where the second number is one.

Give three examples of a unit rate: various...

Why did you choose these examples?
Grade 8 Inquiry Activity

1. At what speed (in miles per hour) was the car traveling if the table below represents distance traveled and time for Juanita on her road trip to Albany?

Remember that \( v = \frac{d}{t} \).

\[
\begin{array}{|c|c|}
\hline
\text{Distance (miles)} & \text{Time (minutes)} \\
\hline
25 & 23 \\
55 & 51 \\
115 & 106 \\
\hline
\end{array}
\]

\[
v = \frac{25 \text{ miles}}{23 \text{ minutes}} \cdot \frac{60 \text{ min}}{1 \text{ hour}}
\]

\[v = 45 \text{ miles per hour}\]

a. Plot the data above on the graph below. (Remember that you cannot control Time so it should be placed on the x-axis as your independent variable.) Don’t forget to label your axes!
b. What is Juanita’s speed during the first 23 minutes of her drive in miles per minute? Draw a triangle on your graph representing this change.

65 mph

c. What is Juanita’s speed during the first 51 minutes of her drive in miles per minute? Draw a triangle on your graph representing this change.

\[
\frac{55 \text{ miles}}{51 \text{ minutes}} = \frac{60 \text{ min}}{1 \text{ hour}} = 64.7 \text{ or } 65 \text{ miles per hour}
\]

d. What is Juanita’s speed during the first 106 minutes of her drive in miles per minute? Draw a triangle on your graph representing this change.

\[
\frac{115 \text{ miles}}{106 \text{ min}} = \frac{60 \text{ min}}{1 \text{ hour}} = 65 \text{ miles per hour}
\]

e. Did you get the same answers for a, b, and c? Explain.

Yes. They traveled at a constant rate.

f. Approximately how many miles can Juanita drive in 1 minute? Round to the nearest mile.

\[
\frac{115 \text{ miles}}{106 \text{ min}} = 1 \text{ mile per minute}
\]

g. Approximately how many miles can Juanita drive in one hour? Round to the nearest mile.

65 miles per hour.
2. The graph below represents how many pages Fred has typed at various points during the day while working on a paper for school. Approximately how many pages is Fred typing every 5 minutes? Draw a triangle on your graph representing this change. What is Fred’s unit rate expressed in pages per minute? Explain any connections you see between the unit rate and the graph below.

\[
\frac{3 \text{ pages}}{10 \text{ min}} = 0.3 \text{ pages/minute}
\]

The unit rate is how much the graph rises (pages typed) for every shift right (minutes passing).
3. This graph represents the number of cookies eaten (y) and the number of people at the table (x). Use this graph to answer all of the questions below.

![Graph showing a line with points at (0,0), (1,3), and (2,6).]

a. Find the unit rate. Draw a triangle on your graph to help you with your calculations. What does the unit rate represent?

\[
\text{unit rate} = \frac{3 \text{ cookies}}{1 \text{ person}} = 3 \text{ cookies per person}
\]

b. What does the point (0, 0) represent? What do the points (1, 3) and (2, 6) represent?

- (0,0) represents no one at the table eating.
- (1,3) represents 1 person eating 3 cookies.
- (2,6) represents 2 people eating 3 cookies each.

c. We can write some of the above values as ratios. 3 cookies to 1 person, or 6 cookies to 2 people. How else can we represent these ratios?

\[
\frac{3}{1}, \frac{6}{2}, 3:1, 6:2
\]
d. Suppose there were 12 people eating a lot of cookies. How many cookies would you expect? How would you write these numbers in a ratio similar to the above ratios? 36 cookies

36:12 or \( \frac{360}{12} \)

3:1 or 3/1

e. How do the above ratios compare to each other? What happens to this ratio as we look at different sized groups of people?

The ratio can be simplified to 3/1 no matter how many people are eating.

Slope: The ratio of vertical to horizontal change

\[
\frac{\text{rise}}{\text{run}}
\]

Slope can be found using this formula:

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]
Find the mistake(s) in the work below. Correct the mistake(s) and explain the error or errors.

1. Find the slope of the line connecting the points (-2, -7) and (3, 5).

\[ m = \frac{-7 - (-7)}{3 - (-2)} = \frac{5 + 7}{3 + 2} = \frac{12}{5} \]

The student forgot to subtract negative values when setting up the problem.

2. Find the slope of the line connecting the points (4, 2) and (2, -4).

\[ m = \frac{-4 - 2}{2 - 4} = \frac{-6}{-2} = \frac{-6}{2} \]

The student subtracted the denominator incorrectly, then divided and got the correct answer by making a mistake with the signs.
Geometry Inquiry Worksheet

1) Graph $y = 3x + 1$ and $y = 3x - 2$ below:

2) Graph $y = -x - 1$ and $y = -x - 4$ below:

3) Graph $y = -\frac{1}{2}x - 3$ and $y = -\frac{1}{2}x + 4$ below:

4) Graph $y = \frac{1}{2}x + 3$ and $y = \frac{1}{2}x - 1$ below:

Name __________________________
5) Summarize any patterns you see above:


6) Discuss this pattern with your group and come to a consensus. Describe your group conclusions:


Before continuing, call your teacher over to discuss your results.

7) Graph \( y = 3x + 1 \) and \( y = -3x - 2 \) below:

![Graph of y = 3x + 1 and y = -3x - 2]

8) Graph \( y = -x - 1 \) and \( y = x - 4 \) below:

![Graph of y = -x - 1 and y = x - 4]
9) Graph $y = -\frac{1}{2}x - 3$ and $y = \frac{1}{2}x + 3$ below:

10) Graph $y = \frac{1}{2}x + 3$ and $y = -\frac{1}{2}x - 1$ below:

11) Summarize any patterns you see above:

12) Discuss this pattern with your group and come to a consensus. Describe your group conclusions:

Before continuing, call your teacher over to discuss your results.
13) You are given the line \( y = 3x + 4 \). Graph the line. Then draw another line parallel to this line. What is the equation of this second line?

![Diagram of a line with equation \( y = 3x + 4 \)]

14) You are given the line \( y = -\frac{1}{2}x + 1 \). Graph the line. Then draw another line perpendicular to this line. What is the equation of this second line?

![Diagram of a line with equation \( y = -\frac{1}{2}x + 1 \)]

Summary:

Parallel lines have slopes that are **the same**.

Perpendicular lines have slopes that are **negative reciprocals**.
Exit Ticket

Name ________________________________

Explain how can you tell if two lines are parallel: ________________________________

How can you tell if two lines are perpendicular: ________________________________

Parallel or Perpendicular? \( y = -3x + 5 \) and \( y = \frac{1}{3}x - 5 \) ________________________________

Parallel or Perpendicular? \( y = 2x - 2 \) and \( y = -\frac{1}{2}x + 2 \) ________________________________
Algebra 2 Inquiry Worksheet

Name ______________________

All questions should be worked in your group. Your goal is discussion that results in consensus. Remember that your calculator output for linear regression gives you “a” and “b” where a represents your slope and b represents your y-intercept for a line in the form $y = ax + b$, which is an alternate form of a slope-intercept equation.

1. The table below presents population census data for the city of Rochester (Census Bureau, 2017).

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980</td>
<td>241,741</td>
</tr>
<tr>
<td>1990</td>
<td>231,636</td>
</tr>
<tr>
<td>2000</td>
<td>219,773</td>
</tr>
<tr>
<td>2010</td>
<td>210,585</td>
</tr>
</tbody>
</table>

a. Using the given data, find the equation of the linear regression line. Using full sentences, explain the meaning of the output values from your calculator. Remember $r$ tells you how closely your equation fits the actual data as well as whether your data trends uphill (for positive $r$ values) or downhill (for negative $r$ values). The closer to 1 or -1, the better the fit.

Equation:

\[
a = -1053.31 \\
b = 2327287.2 \\
r = -0.989246179
\]

$a$ is the population change each year. 
$b$ approximates the population in year 0.
$r$ is negative which tells us the population is decreasing, and the value is very close to -1 so it is a very close approximation of the data.
b. Using your equation, predict the population in the year 2015.

\[ y = -1053.31(2015) + 2,327,287.2 \]
\[ y = 204,867.55 \]

c. The official estimated population in 2015 was 209,802. How would you explain the discrepancy? Use full sentences. Would including this data change your regression equation? How?

The Decline of Kodak, economic changes, societal factors, etc. could all have contributed to a slump in the 1990s & 2000s. Better business could have increased growth after 2000. And so on...

d. Calculate a new regression equation using all of your census data and the 2015 estimated population. How did your equation change? Did it change in the way you expected it to? Explain in full sentences.

\[ y = -957.028x + 21358060.47 \]

The slope is a little less steep reflecting less population loss.
e. What is your new r-value? What does that tell you? Is it different than your old r-value? How? And what does that change mean?

\[ r = -0.9914105128 \]

It is still very close but not quite as close possibly reflecting a little more volatility in the data.

f. What is the unit rate for this data set? Explain what this number means (using full sentences). Why would this be a useful number for city planners to know?

\[ a = -957 \]

This is the unit rate.

On average, Rochester has lost 957 people over the last several decades.

"Yes..."

g. The US Government has census data for the city of Rochester going all the way back to 1820. Could including this information improve your prediction? Why or why not?

Probably not as politics, the economy, etc., have changed drastically since then.
2. The table below presents the average height for children, both boys and girls, of various ages.

<table>
<thead>
<tr>
<th>Age</th>
<th>Average Height (boys)</th>
<th>Average Height (girls)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 years</td>
<td>31 inches</td>
<td>30 inches</td>
</tr>
<tr>
<td>3 years</td>
<td>33 inches</td>
<td>33 inches</td>
</tr>
<tr>
<td>4 years</td>
<td>37 inches</td>
<td>37 inches</td>
</tr>
<tr>
<td>5 years</td>
<td>40 inches</td>
<td>40 inches</td>
</tr>
<tr>
<td>6 years</td>
<td>42 inches</td>
<td>41 inches</td>
</tr>
<tr>
<td>7 years</td>
<td>44 inches</td>
<td>43 inches</td>
</tr>
<tr>
<td>8 years</td>
<td>45 inches</td>
<td>45 inches</td>
</tr>
<tr>
<td>9 years</td>
<td>49 inches</td>
<td>47 inches</td>
</tr>
<tr>
<td>10 years</td>
<td>51 inches</td>
<td>51 inches</td>
</tr>
<tr>
<td>11 years</td>
<td>52 inches</td>
<td>52 inches</td>
</tr>
</tbody>
</table>

a. Find the linear regression equation for the above data. You choose boys or girls.

Boys: \( y = 2.3758x + 26.9576 \), \( r = .9928 \)

Girls: \( y = 2.3697x + 26.49497 \), \( r = .9919 \)

b. What does slope represent in your equation? What does the y-intercept represent? Do you think these are accurate figures?

Slope represents how much an average child grows each year. Y-intercept represents average height at 0 years although this is probably not accurate as babies grow at a different rate.
c. Compare your results with other groups. For groups that used the same
gender, did they get the same equation? If not, troubleshoot. For groups
that found a regression line for the other gender, what equation did they
get? How do the equations for both genders compare?

   Boys start a little taller and grow
   a little faster on average.

d. What is the unit rate for each data set? What does it represent? Explain in
full sentences.

   Boys: 2.3758, Girls: 2.3697

   The unit rate is how much a child
   is expected to grow.

e. Use the above equations to estimate the height of an 18 year old of the
both genders. Do you think these estimate reflect the reality you see
around you at school? Why or why not?

   Girls: 69.15167 or 5'9
   Boys: 69.722 or 5'10

   No. Girls especially seem to slow down growing
during HS.

f. What is your r-value? Interpret your r-value and how it could impact your
answer to part e.

   Boys .9928  Girls r = .9919

   very close to 1.
g. If not, what would help you calculate more accurate figures? What would happen to your r-value if your data were more accurate?

More data could help it be even closer to 1 and more accurate.

3. The data represented in the graph and table below represent the concentration of phenolphthalein as a chemical reaction progresses.

<table>
<thead>
<tr>
<th>Concentration of Phenolphthalein (M)</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00050</td>
<td>0.0</td>
</tr>
<tr>
<td>0.00045</td>
<td>10.5</td>
</tr>
<tr>
<td>0.00040</td>
<td>22.3</td>
</tr>
<tr>
<td>0.00035</td>
<td>35.7</td>
</tr>
<tr>
<td>0.00030</td>
<td>51.1</td>
</tr>
<tr>
<td>0.00025</td>
<td>69.3</td>
</tr>
<tr>
<td>0.00020</td>
<td>91.6</td>
</tr>
<tr>
<td>0.00015</td>
<td>120.4</td>
</tr>
<tr>
<td>0.00010</td>
<td>160.9</td>
</tr>
<tr>
<td>0.000050</td>
<td>230.3</td>
</tr>
<tr>
<td>0.000025</td>
<td>299.6</td>
</tr>
<tr>
<td>0.000015</td>
<td>350.7</td>
</tr>
<tr>
<td>0.000010</td>
<td>391.2</td>
</tr>
</tbody>
</table>

a. Which data points in the table would enable you to calculate a helpful linear regression line to use to predict the concentration of phenolphthalein at 40 seconds? Justify your choice using complete sentences.

The data points near 40 seconds would be best, i.e., 35.7 and 51.1 seconds.
b. Find the equation of a linear regression point using the data you identified in part a.

\[ y = -3.2468 \times 10^{-5} x + .004659 \]

c. Explain the meaning of your slope, your y-intercept, and the unit rate.

Slope is the rate of change of the reaction close to the time we are examining. Y-intercept is a rough estimate of the starting concentration but it is off. The unit rate is the change in concentration per second - same as slope.

d. Graph your line on the graph above. Find another group and compare your line to another group’s graph. Which line would make more accurate predictions? Explain your answer.

e. List the r-values for both your equation and the other group’s equation below. According to r-values, which line is a better fit? Does this agree with your hypothesis from part d above? Why or why not?

Closer r values to 1 or -1 will yield a more accurate regression line equation.
f. The graph below adds a new line on top of our data. This line represents a line tangent to the data at a particular point. This line has the slope that the data would have at the point (51.1, 0.0030) if we zoomed in and were able to find the slope at that particular point. How would predictions using this line compare to predictions using your linear regression line? Discuss in your group. Explain your group’s conclusions below using complete sentences.

It would be very accurate very close to the point of tangency and less accurate further away.
Exit Ticket

Name _______________________

Explain in full sentences the meaning of slope and y-intercept in the context of Linear Regression:

Slope is the rate of change as approximated by our data. Y-intercept reflects a starting value that may or may not be helpful.

Explain in full sentences the role of Unit Rate in the context of Linear Regression:

Unit Rate is the same as slope.

Exit Ticket

Name _______________________

Explain in full sentences the meaning of slope and y-intercept in the context of Linear Regression:

Explain in full sentences the role of Unit Rate in the context of Linear Regression:
Precalculus Inquiry Worksheet

Remember cardinal rule of using transfer paper: **ALWAYS trace and label your axes before moving your transfer paper.** For each problem, first graph the function, and then graph its inverse. To graph the inverse, remember to first find the equation of the inverse by switching $x$ and $y$, solving for $y$, and then rewrite using function notation $f'(x)$. Your job will be to examine the relationship between the two and determine what type of transformations would lead you from the initial function to its inverse. You are also looking for a relationship between the slopes of the original and inverse functions.

1. $f(x) = x - 2$  \hspace{1cm} x = y - 2
   
   Inverse equation: $f'(x) = y = x + 2$
   
   Is the inverse a function? If not, how would you make it so? \textbf{Yes}
   
   Is there a relationship between the slopes of the two sketches? Are they parallel, perpendicular, or neither? \textbf{Parallel}

2. $f(x) = 2$
   
   Inverse equation: $x = 2$
   
   Is the inverse a function? If not, how would you make it so? \textbf{No. It's vertical. Not possible.}
   
   Is there a relationship between the slopes of the two sketches? Are they parallel, perpendicular, or neither? \textbf{Perpendicular}
TEACHING THE STRAND OF CCSS SLOPE

Inverse equation: \( f(x) = 3x + 2 \)  
\[ f^{-1}(y) = \frac{x - 2}{3} = y \]  
\( y < 5 \)

Is the inverse a function? If not, how would you make it so?
Is there a relationship between the slopes of the two sketches? Are they parallel, perpendicular, or neither? Reciprocals. Neither.

4. \( f(x) = \frac{3}{2}x - 4 \)
\[ \frac{3}{2}(x + 4) = y \]
Inverse equation: \( f^{-1}(y) = \frac{3}{2}x + 6 = y \)
Is the inverse a function? If not, how would you make it so? Yes.
Is there a relationship between the slopes of the two sketches? Are they parallel, perpendicular, or neither? Reciprocals. Neither.

5. \( f(x) = -\frac{3}{2}x + 2 \)
\[ x = -\frac{3}{2}y + 2 \]
Inverse equation: \( (x - 2)(-\frac{2}{3}) = y \)
\[ f^{-1}(x) = -\frac{2}{3}y + \frac{10}{3} = y \]
Is the inverse a function? If not, how would you make it so? Yes.
Is there a relationship between the slopes of the two sketches? Are they parallel, perpendicular, or neither? Reciprocals. Neither.

6. What relationship do you see between the graph of a function and its inverse?

Symmetrical. A reflection across \( y = x \)
7. Can you describe another way to transform the original function to the image represented by the inverse using transformations previously studied? Consider horizontal and vertical shifts, dilations, rotations, reflections, alone or in succession. (There may be more than one right answer.)

Reflection across y-axis then 90° rotation clockwise.

Graph the following functions the traditional way and then examine them with your transformation hypothesis from #5 above in mind.

8. \( f(x) = x^2 - 5 \) \quad \begin{equation} x = y^2 - 5 \end{equation}

Inverse equation:

\( \begin{align*} x - 5 &= y^2 \\
\sqrt{x - 5} &= y \\
\end{align*} \)

Is the inverse a function? If not, how would you make it so? **No, restrict the domain so we only consider the positive y.**

Is there a relationship between the slopes of the two sketches? Are they parallel, perpendicular, or neither?

Choose one point on your function, and the inverse of that point. What if you zoomed in and approximated the curve for a small section or used a line tangent to a curve at that point? Slopes would be reciprocals.
TEACHING THE STRAND OF CCSS SLOPE

9. \( f(x) = x^3 + x^2 - 16x - 16 \)
\[ f(x) = (x+1)(x-4)(x+4) \]
Inverse equation:
Beyond the scope of the course.
Is the inverse a function? If not, how would you make it so?
Restrict the domain
Is there a relationship between the slopes of the two sketches? Are they parallel, perpendicular, or neither?

Neither

Choose one point on your function, and the inverse of that point. What if you zoomed in and approximated the curve for a small section or used a line tangent to a curve at that point?

Reciprocals

10. Does your transformation theory from #5 hold true? Do you need to make any adjustments? If you have not yet identified more than one way to use transformations to graph the inverse of a function, do so now.

11. Explain to another student in full sentences a shortcut for sketching the inverse of a function.

Reflect in the line \( y = x \)

OR

Reflect over \( y \)-axis and rotate \( 90^\circ \) clockwise.

12. Summarize the relationships you observed between the slope (or approximate slope) of a curve and the slope of its inverse.

Reciprocals.
Symmetrical.
13. Error Analysis: A student has worked the following problems. Explain the student’s problem solving strategy. Determine whether the student has done the work correctly or incorrectly. If the student has made mistakes, identify and describe the mistakes, and then fix the mistakes. You may use transfer paper.

a. Graph the function and its inverse: \( f(x) = x^3 - 7x + 6 \)

Add synthetic division for 1

\[
\begin{array}{c|ccccc}
1 & 1 & 0 & -7 & 6 \\
1 & 0 & 0 & -6 & 10 \\
\hline
1 & 1 & -6 & 10 \\
\end{array}
\]

Synthetic division helped factor the function to find the roots which provided x-intercepts; the coefficient of the highest power of \( x \) is positive so the graph goes up to the right. Reflecting in \( y = x \), or: in the y-axis followed by 90° rotation clockwise, yields a graph of \( f^{-1}(x) \).
b. Graph the function and its inverse: \( f(x) = \ln(x + 2) \).

Since there are no zeros to solve for, the student in this example graphed the original function and its inverse using their knowledge of transformations. Examine the graph, and draw conclusions about the student’s thought process from the graph.

The graph of \( f(x) \) is the graph of \( \ln x \) translated 2 units left. The student incorrectly graphed \( f^{-1}(x) \) by reflecting \( f(x) \) in the \( x \)-axis instead of in \( y = x \).