GENERALIZED SCHWARZ-PICK ESTIMATES

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Abstract. We obtain higher derivative generalizations of the Schwarz-Pick inequality for analytic self-maps of the unit disk as a consequence of recent characterizations of boundedness and compactness of weighted composition operators between Bloch-type spaces.

1. Introduction

Part of the Schwarz-Pick inequality, sometimes called the invariant Schwarz inequality, says that whenever \( \varphi \) is an analytic self-map of the unit disk \( D \), then

\[
\frac{|\varphi'(z)|(1 - |z|^2)}{1 - |\varphi(z)|^2} \leq 1
\]

for all \( z \) in \( D \). If \( C_\varphi \) is the composition operator defined by \( C_\varphi(f) = f \circ \varphi \) for \( f \) analytic in \( D \), the Schwarz-Pick inequality directly yields the boundedness of all composition operators on the classical Bloch space. We will prove the following generalized Schwarz-Pick estimates.

**Theorem 1.** For \( n \geq 1 \) and \( \varphi \) an analytic self-map of \( D \),

\[
\sup_{z \in D} \frac{|\varphi^{(n)}(z)|(1 - |z|^2)^n}{1 - |\varphi(z)|^2} < \infty.
\]

Our proof of this theorem will be an application of boundedness criteria for weighted composition operators between various Bloch-type spaces recently obtained in [3]. These Bloch-type spaces and boundedness criteria for weighted composition operators will be discussed in the next section, which also contains the proof of the above theorem. A natural generalization of the above result is given in Theorem 3 when \( \varphi \) satisfies an additional condition. In Section 3 we give “little-oh” versions of Theorems 1 and 3 and in Section 4 we briefly discuss converses to our main results.

2. Proof of the main theorem

The Bloch-type spaces we consider here are defined by

\[
B^\alpha = \{ f \text{ analytic in } D : \sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)| < \infty \}.
\]
These become Banach spaces with norms \( |f(0)| + \sup\{(1 - |z|^2)^\alpha|f'(z)| : z \in \mathbb{D}\} \). The range of the parameter \( \alpha \) can be taken to be \( 0 < \alpha < \infty \), although our interest here is restricted to the case \( \alpha \geq 1 \). Note that \( \alpha = 1 \) gives the classical Bloch space \( B \). A weighted composition operator \( uC_\varphi \) is defined for analytic \( u \) on \( \mathbb{D} \) and analytic self-map \( \varphi \) of \( \mathbb{D} \) by \( uC_\varphi(f) = u(f \circ \varphi) \). A characterization of boundedness of \( uC_\varphi \) from \( B^\alpha \) to \( B^\beta \) is given in Theorem 2.1 of [3]; this characterization depends on whether \( 0 < \alpha < 1 \), \( \alpha = 1 \), or \( \alpha > 1 \). Here we will only make use of the \( \alpha > 1 \) case:

**Theorem 2 ([3]).** When \( \alpha > 1 \) and \( \beta > 0 \) the weighted composition operator \( uC_\varphi \) maps \( B^\alpha \) boundedly into \( B^\beta \) if and only if

\[
(a) \quad \sup_{z \in \mathbb{D}} |u(z)| \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\beta} |\varphi'(z)| < \infty \quad \text{and}
\]

\[
(b) \quad \sup_{z \in \mathbb{D}} |u'(z)| \left( \frac{1 - |\varphi(z)|^2}{1 - |\varphi'(z)|^2} \right)^{\alpha-1} < \infty.
\]

Theorem 2 is the key ingredient in our derivation of the generalized Schwarz-Pick estimates. The other ingredient is the observation that since \( f \in B^\alpha \) if and only if \( f' \in B^{\alpha+1} \), and all composition operators are bounded from \( B^1 \) to \( B^1 \), it follows that the operators \( D^nC_\varphi \) are bounded from \( B^1 \) to \( B^{n+1} \) for all \( n \geq 1 \) and all \( \varphi \), where \( D^n \) denotes the \( n^{th} \) derivative operator.

**Proof of Theorem 2** For \( n = 1 \), the result is the classical Schwarz-Pick inequality. The rest of the argument proceeds by induction, however it is instructive to look explicitly at the \( n = 2 \) case. For this, note that \( DC_\varphi \) is bounded from \( B^1 \) to \( B^2 \) for all \( \varphi \), as noted above. We have \( DC_\varphi(f) = (f' \circ \varphi) \varphi' \). Thus the weighted composition operator \( \varphi'C_\varphi \) is bounded from \( B^2 \) to \( B^2 \), since \( f \in B^1 \) if and only if \( f' \in B^2 \). In particular by (b) of the boundedness criteria above we have the desired statement for \( n = 2 \).

Now fix an integer \( n \geq 2 \) and assume by induction that the generalized Schwarz-Pick estimates hold for all positive integers less than or equal to \( n \). We will show that the estimate holds for \( n + 1 \). Consider the bounded operator \( D^nC_\varphi : B^1 \to B^{n+1} \). If we can show that \( \varphi''(n)C_\varphi \) is bounded from \( B^2 \) to \( B^{n+1} \), then again part (b) of the boundedness criteria above will yield the generalized Schwarz-Pick estimate for \( n + 1 \). To see why the boundedness of \( \varphi''(n)C_\varphi : B^2 \to B^{n+1} \) follows from the boundedness of \( D^nC_\varphi : B^1 \to B^{n+1} \) we consider the expansion of \( D^n(f \circ \varphi) = (f \circ \varphi)^{(n)} \) by Faa di Bruno’s formula (see, for example, [4]):

\[
(f \circ \varphi)^{(n)}(z) = \sum_{k_1 + k_2 + \cdots + k_n = n} \frac{n!}{k_1!k_2! \cdots k_n!} f^{(k)}(\varphi(z)) \prod_{j=1}^n \left( \frac{\varphi^{(j)}(z)}{j!} \right)^{k_j},
\]

where \( k = k_1 + k_2 + \cdots + k_n \) and this sum is over all non-negative integers \( k_1, k_2, \ldots, k_n \) satisfying \( k_1 + 2k_2 + \cdots + nk_n = n \). In particular, one of the terms of this sum is \( f'(\varphi(z)) \varphi''(n)(z) \) and the remaining terms involve products of \( f^{(k)} \circ \varphi(z) \) (\( 1 < k \leq n \)) with products of derivatives of \( \varphi \). Writing Faa di Bruno’s formula in operator notation we have

\[
(1) \quad D^nC_\varphi = \sum_{k_1 + k_2 + \cdots + k_n = n} \frac{n!}{k_1!k_2! \cdots k_n!} \prod_{j=1}^n \left( \frac{D^{j} \varphi}{j!} \right)^{k_j} C_\varphi D^k.
\]
With $k_n = 1$ (and therefore also $k_1 = k_2 = \cdots = k_{n-1} = 0$) we obtain on the right the term $\varphi^{(n)} C_{\varphi} D$. If $k_n = 0$ we obtain (constant multiples of) the terms

$$
\prod_{j=1}^{n-1} (\varphi^{(j)})^{k_j} C_{\varphi} D^k,
$$

where $k = k_1 + \cdots + k_{n-1}$, and $k_1 + 2k_2 + \cdots + (n-1)k_{n-1} = n$. Set

$$
u(z) = \prod_{j=1}^{n-1} (\varphi^{(j)}(z))^{k_j},
$$

(2)

where the non-negative integers $k_1, \ldots, k_{n-1}$ are as just described. Our goal is to show that each weighted composition operator $u \cdot \varphi C_{\varphi}$ is bounded from $B^{k+1}$ to $B^{n+1}$; this together with the boundedness of $D^n C_{\varphi} : B^1 \to B^{n+1}$ will imply the boundedness of $\varphi^{(n)} C_{\varphi}$ from $B^2$ to $B^{n+1}$. To show that $u \cdot \varphi C_{\varphi}$ is bounded from $B^{k+1}$ to $B^{n+1}$ we must verify conditions (a) and (b) of Theorem 2.

For condition (a) we observe that the product

$$
|u(z)| \leq \frac{(1 - |z|^2)^{n+1}}{(1 - |\varphi(z)|^2)^{k+1}} |\varphi'(z)|
$$

can be written as

$$
\left(\frac{(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2}\right) \prod_{j=2}^{n-1} \left(\frac{(1 - |z|^2)|\varphi^{(j)}(z)|}{1 - |\varphi(z)|^2}\right)^{k_j},
$$

(3)

since $n+1 = (k_1 + 1) + 2k_2 + \cdots + (n-1)k_{n-1}$. Using the induction hypothesis we see that

$$
\sup_{z \in B} \frac{(1 - |z|^2)^{n+1}}{(1 - |\varphi(z)|^2)^{k+1}} |u(z)| |\varphi'(z)| < \infty.
$$

For condition (b) of Theorem 2 notice that

$$
u'(z) = \sum_{i=1}^{n-1} k_i \varphi^{(i)}(z) \prod_{j=1, j \neq i}^{n-1} (\varphi^{(j)}(z))^{k_j}.
$$

(4)

We claim that

$$
\sup_{z \in B} \frac{(1 - |z|^2)^{n+1}}{(1 - |\varphi(z)|^2)^{k+1}} |\nu'(z)| < \infty.
$$

To see this, note that when $k_i \neq 0$ we see by the induction hypothesis that

$$
|\varphi^{(i)}(z)|^{k_i} |\varphi^{(i+1)}(z)| \prod_{j=1, j \neq i}^{n-1} |\varphi^{(j)}(z)|^{k_j}
$$

is bounded above by a constant multiple of

$$
\frac{(1 - |\varphi(z)|^2)^{k_i}}{(1 - |z|^2)^{k_i(k_i - 1)}} \prod_{j=1, j \neq i}^{n-1} \left(\frac{1 - |\varphi(z)|^2}{1 - |z|^2}\right)^{k_j} = \frac{(1 - |\varphi(z)|^2)^k}{(1 - |z|^2)^{n+1}},
$$

(5)

and our claim follows.
Conditions (a) and (b) in Theorem 2 are satisfied and the operator $uC_{\varphi}$ maps $B^{k+1}$ boundedly into $B^{n+1}$. The operator $D^k$ maps $B^1$ boundedly onto $B^{k+1}$. Thus, for each $k$ as above with $k_n = 0$, the operator

$$
\prod_{j=1}^{n-1} \left( \varphi^{(j)} \right)^{k_j} C_{\varphi} D^k
$$

maps $B^1$ boundedly into $B^{n+1}$. We conclude that $\varphi^{(n)} C_{\varphi} D$ maps $B^1$ boundedly into $B^{n+1}$. Since $D$ maps $B^1$ onto $B^2$, the weighted composition operator $\varphi^{(n)} C_{\varphi}$ maps $B^2$ boundedly onto $B^{n+1}$. By condition (b) of Theorem 2 this implies that

$$
|\varphi^{(n+1)}(z)| \frac{(1 - |z|^2)^{n+1}}{1 - |\varphi(z)|^2} = |(\varphi^{(n)})'(z)| \frac{(1 - |z|^2)^{n+1}}{1 - |\varphi(z)|^2}
$$

is bounded. This completes the induction, and the proof.

Theorem 3 can easily be generalized as follows.

**Theorem 3.** Let $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be an analytic map such that

$$
\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha} < \infty
$$

for some $\alpha, \beta > 0$. Then for each integer $n \geq 2$,

$$
\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{\beta + n - 1} |\varphi^{(n)}(z)|}{(1 - |\varphi(z)|^2)^\alpha} < \infty.
$$

**Proof.** The hypothesis insures that $C_{\varphi}$ is bounded from $B^{\alpha}$ to $B^{\beta}$ (3, Corollary 2.4) so $DC_{\varphi}$ is bounded from $B^{\alpha}$ to $B^{\beta+1}$. Since $DC_{\varphi} = \varphi'C_{\varphi}D$ it follows that $\varphi'C_{\varphi}$ must be bounded from $B^{\alpha+1}$ to $B^{\beta+1}$. Part (b) of Theorem 2 gives the desired conclusion for $n = 2$. We proceed by induction in much the same way as was done in the proof of Theorem 1. Assume the result holds for all positive integers less than or equal to $n$. To obtain the result for $n + 1$ we show that $\varphi^{(n)} C_{\varphi}$ is bounded from $B^{\alpha+1}$ to $B^{\beta+n}$, and then appeal to Theorem 2. As in the proof of Theorem 1, boundedness of $\varphi^{(n)} C_{\varphi}$ will follow from the boundedness of $D^\alpha C_{\varphi}$ from $B^{\alpha}$ to $B^{\beta+n}$ and (1) if we can show that $uC_{\varphi}$ is bounded from $B^{\alpha+k}$ to $B^{\beta+n}$, $1 \leq k < n$, when $u$ is given by (2). Condition (a) of Theorem 2 follows from the observation that

$$
|u(z)| \frac{(1 - |z|^2)^{\beta+n}}{(1 - |\varphi(z)|^2)^{\alpha+k}} |\varphi'(z)| = \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha} \prod_{j=1}^{n-1} \left( \frac{(1 - |z|^2)^{\beta} |\varphi^{(j)}(z)|}{1 - |\varphi(z)|^2} \right)^{k_j}.
$$

The first factor is bounded on $\mathbb{D}$ by hypothesis, and the other factors are bounded on $\mathbb{D}$ by Theorem 1.

Similarly, to check condition (b) we must show that

$$
\frac{|u'(z)|(1 - |z|^2)^{\beta+n}}{(1 - |\varphi(z)|^2)^{\alpha+k-1}}
$$
is bounded on $\mathbb{D}$. Using the expression for $u'(z)$ given in (4) this follows by observing that for $k_i \geq 1$ the expression
\[
|\varphi^{(i)}(z)|^{k_i - 1} |\varphi^{(i+1)}(z)| \prod_{j=1, j \neq i}^{n-1} |\varphi^{(j)}(z)|^{k_j}
\]
is bounded above by a constant multiple of
\[
\frac{(1 - |\varphi(z)|^2)^\alpha}{(1 - |z|^2)^{\beta+1}} \prod_{j=1, j \neq i}^{n-1} \left( \frac{1 - |\varphi(z)|^2}{(1 - |z|^2)^{\beta+1}} \right)^{k_j} = \frac{(1 - |\varphi(z)|^2)^{\alpha+k-1}}{(1 - |z|^2)^{\beta+n}},
\]
which gives the desired result. This completes the verification of the boundedness of $uC_\varphi$ from $B_0^{\alpha+k}$ to $B_0^{\beta+n}$, and the theorem follows exactly as in Theorem 1.

3. The hyperbolic little Bloch class

Recall that an analytic self-map of the disk $\varphi$ is said to be in the hyperbolic little Bloch class $B_h^0$ if
\[
\lim_{|z| \to 1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.
\]
Note this implies in particular that $\varphi$ is in the little Bloch space $B_0^0$, the subspace of $B$ consisting of Bloch functions $f$ satisfying $\lim_{|w| \to 1} |f'(w)|(1 - |w|^2) = 0$. The hyperbolic little Bloch class appears in the characterization of those composition operators which are compact on the little Bloch space: $C_\varphi$ is compact from $B_0$ to itself if and only if $\varphi \in B_h^0$ (2, Theorem 1).

A particular case of the next result shows that functions in the hyperbolic little Bloch class satisfy a little-oh version of our generalized Schwarz-Pick estimates.

Theorem 4. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map such that
\[
\lim_{|z| \to 1} \frac{(1 - |z|^2) |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha} = 0
\]
for some $\alpha, \beta > 0$. Then for each integer $n \geq 2$,
\[
\lim_{|z| \to 1} \frac{(1 - |z|^2)^{\beta+n-1} |\varphi^{(n)}(z)|}{(1 - |\varphi(z)|^2)^\alpha} = 0.
\]
In particular, if $\varphi \in B_h^0$, then
\[
\lim_{|z| \to 1} \frac{(1 - |z|^2)^n |\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} = 0
\]
for every positive integer $n$.

Theorem 4 can be proved by similar techniques to those employed in Theorem 3 using Theorem 3.1 of [3] which characterizes compactness of weighted composition operators from $B_0^\alpha$ to $B_0^\beta$ by little oh analogues of (a) and (b) of Theorem 2. We omit the details.
4. Converse results

For certain positive $\alpha$ and $\beta$ the implications in Theorem 3 and Theorem 4 are actually logical equivalences.

**Theorem 5.** Let $\varphi$ be an analytic self-map of the unit disk and $\beta > \alpha > 0$. Then

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{\beta} |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\alpha}} < \infty$$

if and only if

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{\beta+n-1} |\varphi^{(n)}(z)|}{(1 - |\varphi(z)|^2)^{\alpha}} < \infty$$

for each positive integer $n$.

Furthermore,

$$\lim_{|z| \to 1^-} \frac{(1 - |z|^2)^{\beta} |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\alpha}} = 0$$

if and only if

$$\lim_{|z| \to 1^-} \frac{(1 - |z|^2)^{\beta+n-1} |\varphi^{(n)}(z)|}{(1 - |\varphi(z)|^2)^{\alpha}} = 0$$

for each positive integer $n$.

We do not give the proof of this result here, but note that the interest in the first part of Theorem 5 in the “if” direction is when $0 < \alpha < \beta < 1$, as the condition

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{\beta} |\varphi'(z)|}{1 - |\varphi(z)|^2} < \infty$$

holds automatically for all self-maps when $\alpha \leq \beta$ and $\beta \geq 1$.

The “if” directions of the two statements in Theorem 5 need not hold if $\beta < \alpha$. For example, if $n \geq 2$ and $\varphi(z) = \frac{1}{2} z^{n-1} + \frac{1}{2}$, then $\varphi^{(n)}(z) = 0$ so that

$$\sup_{z \in \mathbb{D}} \frac{|\varphi^{(n)}(z)|(1 - |z|^2)^{\beta+n-1}}{(1 - |\varphi(z)|^2)^{\alpha}} = 0$$

for any $\alpha, \beta > 0$. However if we consider $z = r \in (0, 1)$ we have that

$$\frac{(1 - r^2)^{\beta} |\varphi'(r)|}{(1 - |\varphi(r)|^2)^{\alpha}}$$

is unbounded as $r \to 1^-$ if $\beta < \alpha$, and tends to a finite positive constant as $r \to 1^-$ if $\beta = \alpha$. Thus the hypothesis $\alpha < \beta$ is sharp for the second statement in Theorem 5 and close to sharp for the first statement.
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