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TOEPLITZ OPERATORS ON BLOCH-TYPE SPACES

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ABSTRACT. We characterize complex measures μ on the unit disk for which the Toeplitz operator T_μ^α, $\alpha > 0$, is bounded or compact on the Bloch type spaces B^α.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{D}$ be the unit disk on the complex plane. Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on $\mathbb{D}$.

For a complex measure μ, $\alpha > 0$, and $b \in L^1$, define a Toeplitz operator as follows:

$$T_\mu^\alpha(b)(z) = \alpha \int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha-1}b(w)}{(1-\bar{w}z)^{\alpha+1}}\, d\mu(w).$$

We also define the general Bergman projection of the measure μ and for $\alpha > -1$ as follows:

$$P_\alpha(\mu)(z) = (\alpha+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha}}{(1-\bar{w}z)^{\alpha+2}}\, d\mu(w).$$

The general Bergman projection of the function b is

$$P_\alpha(b)(z) = (\alpha+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha}b(w)}{(1-\bar{w}z)^{\alpha+2}}\, dA(w).$$

Thus $T_\mu^\alpha(b)(z) = P_{\alpha-1}(\mu_b)(z)$, where $d\mu_b(z) = b(z)d\mu(z)$. Note that these choices of the indexes provide the standard notation for the general Bergman projections and for the standard case $\alpha = 1$ as follows:

When $\alpha = 1$ and the measure μ is such that $d\mu(z) = f(z)dA(z)$, with $f \in L^1$, we have that T_μ^1 is the standard Toeplitz operator defined by

$$T_\mu^1(b) = T_f(b) = P(fb), \quad \forall b \in L^1.$$

Here $P = P_0$ denotes the standard Bergman projection. Recall that if f is in L^∞, then T_f is bounded on the Bergman spaces L_a^p, $p > 1$.

Toeplitz operators have been studied extensively on the Bergman spaces by many authors. For references, see for example [9]. In this paper we study the boundedness and compactness of general Toeplitz operators on the α-Bloch spaces.

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For $\alpha > 0$, the *α-Bloch spaces* B^α are the spaces of analytic functions f on $\mathbb{D}$ such that

$$M^\alpha(f) = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^\alpha < \infty.$$

Each B^α is a Banach space with norm of f equal to $\|f\|_{B^\alpha} = M^\alpha(f) + |f(0)|$.

An analytic function on $\mathbb{D}$ belongs to the *little α-Bloch space* B_0^α, $\alpha > 0$, whenever

$$\lim_{|z|\to 1} |f'(z)|(1-|z|^2)^\alpha = 0.$$

The spaces B_0^α are the subspaces of B^α that are the closure of the polynomials with respect to the B^α norm.

The α-Bloch spaces and operators on them have been studied in many different contexts. For more references and details on the next few facts stated below see [9] and [10].

For $\alpha = 1$, $B^1 = B$ is the classical Bloch space. There is a natural connection between the Bloch space and the Toeplitz operators via the fact that $P(L^\infty) = B$. A necessary condition for boundedness of T_f on B is that $Pf \in B$, and so the condition is satisfied whenever $f \in L^\infty$. As we will see later, this is not a sufficient condition for boundedness of T_f on B.

In general, the growth condition of a function f in B^α with $f(0) = 0$ is determined by $|f(z)| \le \|f\|_{B^\alpha} \int_0^1 \frac{|z|dt}{(1-|z|t)^\alpha}$. Thus, for $\alpha = 1$, we get that

$$|f(z)| \le \|f\|_B \log \frac{1}{1-|z|},$$

while for $\alpha > 1$ we have that $|f(z)| \le \frac{1}{\alpha-1}\|f\|_{B^\alpha}\frac{1}{(1-|z|)^{\alpha-1}}$. For $0 < \alpha < 1$, the spaces B^α are Lipschitz spaces $Lip_{1-\alpha}$ and $B^\alpha \subset H^\infty$.

For $0 < \alpha < \beta$, we have that $B^\alpha \subset B^\beta$. The Bloch space B is included in all Bergman spaces L_a^p, $p \ge 1$, but for large α, such as $\alpha \ge 2$, B^α gets so large that, for example, it includes the Bergman space L_a^2.

In presenting our results we will also refer to the logarithmic Bloch space.

The *logarithmic Bloch space* LB is the space of analytic functions f on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)\log\frac{2}{1-|z|^2} < \infty.$$

Correspondingly, the *little logarithmic Bloch space* LB_0 is the space of analytic functions f on $\mathbb{D}$ such that

$$\lim_{|z|\to 1} |f'(z)|(1-|z|^2)\log\frac{2}{1-|z|^2} = 0.$$

Throughout the paper, in order to have the operator $T_\mu^\alpha, \alpha > 0$ well defined, we will assume that the complex measure μ is such that, for all g in B^α

$$\int_{\mathbb{D}} |g(w)|(1-|w|^2)^{\alpha-1}|d\mu(w)| < \infty.$$

Note that in the case when $d\mu(z) = f(z)dA(z)$, a sufficient condition on μ, so that the integral above is finite, is that $f \in L^1$, when $\alpha > 1$, $f \in L^1(\log\frac{1}{1-|w|^2}dA(w))$, when $\alpha = 1$ and $f \in L^1((1-|w|^2)^{\alpha-1}dA(w))$, when $0 < \alpha < 1$.

2. Bounded Toeplitz operators

In this section we will present our main characterization of bounded Toeplitz operators on general α-Bloch spaces. The result includes a restriction on the inducing measure that is a generalization of several known cases.

For a complex measure μ and $\alpha > 0$, we will say that μ satisfies the condition R_α if

$$R_\alpha(\mu)(w) = \alpha(1-|w|^2)\int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha-1}}{\overline{(z-w)}(1-w\bar{z})^{\alpha+1}}\, d\mu(z) \in L^\infty.$$

We get another form of R_α using the identity $(1-|w|^2) = 1 - w\bar{z} + w(\bar{z}-\bar{w})$:

$$R_\alpha(\mu)(w) = \alpha\int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha-1}}{\overline{(z-w)}(1-w\bar{z})^{\alpha}}\, d\mu(z) + wP_{\alpha-1}(\mu)(w).$$

The special case when the measure μ is such that $d\mu(z) = f(z)dA(z)$, with $f \in H^\infty$, is the case when the Toeplitz operator is a multiplication operator. For $f \in B^\alpha$, as we will see below, μ satisfies the condition R_α if and only if $f \in H^\infty$.

Arazy's ([1]) and Zhu's ([8] and [10]) results provide a complete characterization of bounded multiplication operators on α-Bloch spaces. We will state them in terms of multipliers.

Theorem A ([10]). *Let $M(X)$ denote the space of multipliers of the Banach space of functions X, i.e., $M(X) = \{f \in X \ :\ fg \in X,\ \forall g \in X\ \}$.*

(i) *If $0 < \alpha < 1$, then $M(B^\alpha) = B^\alpha$ and $M(B_0^\alpha) = B_0^\alpha$.*
(ii) *If $\alpha = 1$, then $M(B^\alpha) = M(B_0^\alpha) = H^\infty(\mathbb{D}) \cap LB$.*
(iii) *If $\alpha > 1$, then $M(B^\alpha) = M(B_0^\alpha) = H^\infty(\mathbb{D})$.*

As a consequence of the above theorem we can see that, for example, not every bounded function f induces a bounded Toeplitz operator T_f on the Bloch space B. As we will see later, for $f \in L^\infty$, T_f is bounded on B if and only if $Pf \in LB$, which is a generalization of the analytic case of the above theorem. We will also see that there exist unbounded L^1 functions that induce bounded Toeplitz operators on the Bloch spaces.

In the next proposition we give more details about the condition R_α in some special cases.

Proposition 2.1. *Let μ be a measure such that $d\mu(z) = f(z)dA(z)$, with $f \in L^1$. Then:*

(i) *If $f \in L^\infty$, μ satisfies the condition R_α, $\alpha > 0$.*
(ii) *If f is analytic and in B^α, then $R_\alpha(\mu)(w) = wf(w)$.*
(iii) *If f is conjugate analytic, $\bar{f}$ is in B^α and $f(0) = 0$, then $R_\alpha(\mu)(w) = \frac{1}{\bar{w}}f(w)$.*

Proof. Let $\psi_w(z) = \frac{w-z}{1-\bar{w}z}$, for z, w in $\mathbb{D}$.

(i) For $\alpha > 0$ and $f \in L^\infty$,

$$\begin{aligned} |R_\alpha(\mu)(w)| &= \alpha(1-|w|^2)\left|\int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha-1}\bar{f}(z)}{(z-w)(1-\bar{w}z)^{\alpha+1}}\, dA(z)\right| \\ &\le \alpha\|f\|_\infty(1-|w|^2)\int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha-1}}{|z-w||1-\bar{w}z|^{\alpha+1}}\, dA(z) \\ &\le c\alpha\|f\|_\infty(1-|w|^2)(1-|w|^2)^{-1} = c\alpha\|f\|_\infty. \end{aligned}$$

The second inequality follows from the fact that $\int_{\mathbb{D}} \frac{(1-|z|^2)^\beta}{|z-w|\ |1-\bar{w}z|^\gamma} dA(z)$ is equivalent to $(1-|w|^2)^{\beta-\gamma+1}$, whenever $\beta-\gamma+1<0$, which can be easily derived from the well-known Forelli-Rudin estimates (see [6], page 17, or [8], page 53). The half-plane version of the proof of this equivalency can be found in [7].

Thus $R_\alpha(\mu)(w) \in L^\infty$, and (i) is proved.

(ii) Let f be in B^α. With a change of variable $z=\psi_w(u)$, we get that

$$\begin{aligned} R_\alpha(\mu)(w) &= \alpha \int_{\mathbb{D}} \frac{(1-|\psi_w(u)|^2)^{\alpha-1}|\psi_w'(u)|^2}{-\bar{u}(1-w\bar{\psi}_w(u))^{\alpha+2}} f\circ\psi_w(u)dA(u) + wP_{\alpha-1}(f)(w) \\ &= -\alpha \int_{\mathbb{D}} \frac{(1-|u|^2)^{\alpha-1}}{\bar{u}(1-\bar{w}u)^{\alpha+1}} f\circ\psi_w(u)dA(u) + wf(w) = I + wf(w). \end{aligned}$$

For a fixed w we have that $\frac{f\circ\psi_w(u)}{(1-\bar{w}u)^{\alpha+1}}$ belongs to B^α whenever $f\in B^\alpha$, and so $\frac{(1-|u|^2)^{\alpha-1} f\circ\psi_w(u)}{(1-\bar{w}u)^{\alpha+1}}$ must be in L^1. Using Taylor series expansion it is not hard to see that then $I=0$. Hence $R_\alpha(\mu)(w) = wf(w)$.

(iii) Let f be conjugate analytic and let $f(0)=0$. Then $P_{\alpha-1}(f)(w)=0$, and as in the proof of part (ii), we have that

$$\begin{aligned} \overline{R_\alpha(\mu)(w)} &= -\alpha \int_{\mathbb{D}} \frac{(1-|u|^2)^{\alpha-1}}{u(1-\bar{u}w)^{\alpha+1}} \overline{f\circ\psi_w}(u)dA(u) \\ &= -\alpha \int_{\mathbb{D}} \frac{(1-|u|^2)^{\alpha-1}}{(1-\bar{u}w)^{\alpha+1}} \frac{\overline{f\circ\psi_w}(u) - \overline{f}(w)}{u} dA(u), \end{aligned}$$

since $\int_{\mathbb{D}} \frac{(1-|u|^2)^{\alpha-1}}{u(1-\bar{u}w)^{\alpha+1}} dA(u) = 0$. Using the fact that $\frac{\overline{f\circ\psi_w}(u)-\overline{f}(w)}{u}$ is an analytic integrable function, we get that

$$\overline{R_\alpha(\mu)(w)} = -\frac{\overline{f\circ\psi_w}(w) - \overline{f}(w)}{w} = -\frac{\bar{f}(0)-\bar{f}(w)}{w} = \frac{1}{w}\bar{f}(w).$$

□

Remark. If f from the previous proposition is harmonic, then $R_\alpha(\mu)$ is also harmonic. If also f is a harmonic extension of an $L^1(T)$ function, then $R_\alpha(\mu)(e^{i\theta}) = e^{i\theta} f(e^{i\theta})$. Thus, for f harmonic and in L^∞, we have that $R_\alpha(\mu)$ is also harmonic and in L^∞, with $\|R_\alpha(\mu)\|_\infty = \|f\|_\infty$.

We will use the following lemma in the proof of our next theorem. The result is known in a more general form and can be found, for example, in [2]. We include a proof for completeness.

Lemma 2.1. *Let $h \in L^1_a$ and $g\in B^\alpha$, $\alpha>0$. Then we have that*

$$H_g^\alpha h(z) = (I-P_\alpha)(\bar{g}h)(z) = \int_{\mathbb{D}} \frac{\overline{g'(w)}\ h(w)\ (1-|w|^2)^{\alpha+1}}{(z-w)(1-\bar{w}z)^{\alpha+1}} dA(w).$$

Proof. We will use the following three formulas which can be found in [3] and [10]. For $g \in B^\alpha$ and $h \in L^1_a$:

$$\text{(a)} \quad H^\alpha_{\bar g} h(z) = (\alpha+1)\int_{\mathbb{D}} \frac{(\bar g(z) - \bar g(u))\ (1-|u|^2)^\alpha}{(1-\bar u z)^{\alpha+2}} h(u)dA(u),$$

$$\text{(b)} \quad g(z) = g(0) + \int_{\mathbb{D}} \frac{(1-|u|^2)^{\alpha+1} g'(u)}{\bar u(1-\bar u z)^{\alpha+2}} dA(u),$$

$$\text{(c)} \quad \left(\left(H^\alpha_{\bar g}\ h\right)\circ\psi_z\right)(\psi_z')^{\frac{\alpha+2}{2}} = H^\alpha_{\bar g\circ\psi_z}\ \left((h\circ\psi_z)(\psi_z')^{\frac{\alpha+2}{2}}\right).$$

Without loss of generality, let us assume that $g(0)=0$. Then

$$\begin{aligned}
H^\alpha_{\bar g} h(0) &= (\alpha+1)\int_{\mathbb{D}} (-\bar g(u))(1-|u|^2)^\alpha h(u)dA(u) \\
&= -(\alpha+1)\int_{\mathbb{D}} \overline{\left(\int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha+1}\ g'(w)}{\bar w(1-\bar w u)^{\alpha+2}} dA(w)\right)}(1-|u|^2)^\alpha h(u)dA(u) \\
&= -(\alpha+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha+1}\ \overline{g'}(w)}{w}\left(\int_{\mathbb{D}} \frac{(1-|u|^2)^\alpha\ h(u)}{(1-\bar u w)^{\alpha+2}} dA(u)\right) dA(w) \\
&= -\int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha+1}\ \overline{g'}(w)}{w} h(w)dA(w).
\end{aligned}$$

Since $1-|\psi_z(v)|^2 = \left(1-|v|^2\right)|\psi_z'(v)|$ and $\psi_z'(v) = \frac{|z|^2-1}{(1-\bar z v)^2} = \frac{1}{\psi_z'(\psi_z(v))}$, we have:

$$\begin{aligned}
H^\alpha_{\bar g} h(z) &= \frac{1}{(|z|^2-1)^{\frac{\alpha+2}{2}}}\left(\left(H^\alpha_{\bar g} h\right)\circ\psi_z\right)(0)\ (\psi_z'(0))^{\frac{\alpha+2}{2}} \\
&= \frac{1}{(|z|^2-1)^{\frac{\alpha+2}{2}}}\left(H^\alpha_{\bar g\circ\psi_z}\left((h\circ\psi_z)(\psi_z')^{\frac{\alpha+2}{2}}\right)\right)(0) \\
&= \frac{-1}{(|z|^2-1)^{\frac{\alpha+2}{2}}}\int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha+1}\ \overline{(g\circ\psi_z)'}(w)}{w} h\circ\psi_z(w)(\psi_z'(w))^{\frac{\alpha+2}{2}} dA(w) \\
&= -\int_{\mathbb{D}} \frac{\overline{g'}(v)\ h(v)\ (1-|v|^2)^{\alpha+1}(1-|z|^2)(1-\bar z v)(1-\bar v z)^2}{(z-v)(1-\bar v z)^{\alpha+2}(|z|^2-1)|1-\bar z v|^2} dA(v) \\
&= \int_{\mathbb{D}} \frac{\overline{g'}(v)\ h(v)\ (1-|v|^2)^{\alpha+1}}{(z-v)(1-\bar v z)^{\alpha+1}} dA(v).
\end{aligned}$$

□

We have the following theorem:

Theorem 2.1. *Suppose μ satisfies the condition R_α, i.e.,*

$$(1) \qquad R_\alpha(\mu)(w) = \alpha(1-|w|^2)\int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha-1}}{\overline{(z-w)}(1-w\bar z)^{\alpha+1}}\, d\mu(z) \in L^\infty.$$

Then we have:

(i) *If $0<\alpha<1$, then T^α_μ is bounded on B^α if and only if $P_{\alpha-1}(\mu)\in B^\alpha$.*
(ii) *If $\alpha=1$, then T^α_μ is bounded on B^α if and only if $P_{\alpha-1}(\mu)\in LB$.*
(iii) *If $\alpha>1$, then T^α_μ is bounded on B^α if and only if $P_{\alpha-1}(\mu)\in B$.*

Proof. Let $g\in B^\alpha$ and $h\in L^1_a$. It is known that $(L^1_a)^* = B^\alpha$ under the integral pairing $\langle h,g\rangle_\alpha = \alpha\int_{\mathbb{D}} h(z)\bar g(z)(1-|z|^2)^{\alpha-1}\, dA(z)$. See [10, Theorem 14] for more details.

Thus, by Fubini's Theorem,

$$\begin{aligned}\langle h, T_\mu^\alpha(g)\rangle_\alpha &= \alpha\int_{\mathbb{D}} h(z)\overline{T_\mu^\alpha(g)(z)}(1-|z|^2)^{\alpha-1}\,dA(z)\\ &= \alpha\int_{\mathbb{D}} h(z)\overline{g(z)}(1-|z|^2)^{\alpha-1}\,d\bar\mu(z)\\ &= \alpha\int_{\mathbb{D}} P_\alpha(h\bar g)(z)(1-|z|^2)^{\alpha-1}\,d\bar\mu(z)\\ &\quad + \alpha\int_{\mathbb{D}}[(I-P_\alpha)(h\bar g)](z)(1-|z|^2)^{\alpha-1}\,d\bar\mu(z) = I_1 + I_2.\end{aligned}$$

From Lemma 2.1 we have that

$$[(I-P_\alpha)(h\bar g)](z) = \int_{\mathbb{D}} \frac{\overline{g'(w)}h(w)(1-|w|^2)^{\alpha+1}}{(z-w)(1-\bar w z)^{\alpha+1}}dA(w).$$

Hence, by (1),

$$\begin{aligned}|I_2| &= \left|\alpha\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{\overline{g'(w)}h(w)(1-|w|^2)^{\alpha+1}}{(z-w)(1-\bar w z)^{\alpha+1}}\,dA(w)(1-|z|^2)^{\alpha-1}\,d\bar\mu(z)\right|\\ &= \left|\int_{\mathbb{D}} \overline{g'(w)}h(w)(1-|w|^2)^{\alpha}\overline{R_\alpha(\mu)(w)}\,dA(w)\right|\\ &\le \|g\|_{B^\alpha}\|h\|_1\|R_\alpha(\mu)\|_\infty \le C < \infty.\end{aligned}$$

On the other hand, by Fubini's Theorem,

$$I_1 = \int_{\mathbb{D}} h(w)\overline{g(w)Q_\alpha(\mu)(w)}(1-|w|^2)^\alpha\,dA(w),$$

where $Q_\alpha(\mu)(w) = \alpha(\alpha+1)\int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha-1}}{(1-\bar z w)^{\alpha+2}}d\mu(z)$.

Hence $T_\mu^\alpha(g) \in B^\alpha$ if and only if

$$(1-|w|^2)^\alpha g(w)Q_\alpha(\mu)(w) \in L^\infty. \tag{2}$$

The relation between $Q_\alpha(\mu)$ and $P_{\alpha-1}(\mu)$ is

$$Q_\alpha(\mu)(w) = (\alpha+1)P_{\alpha-1}(\mu)(w) + wP'_{\alpha-1}(\mu)(w). \tag{3}$$

It is easy to see that $P_{\alpha-1}(\mu)$ is in B^α, LB and B as $0<\alpha<1$, $\alpha=1$ and $\alpha>1$, respectively, implies that $Q_\alpha(\mu)$ satisfies, respectively:

$$\begin{aligned}&Q_\alpha(\mu)(w)(1-|w|^2)^\alpha \in L^\infty,\ \text{if } 0<\alpha<1;\\ &Q_\alpha(\mu)(w)(1-|w|^2)\log\frac{2}{1-|w|^2} \in L^\infty,\ \text{if } \alpha=1;\\ &Q_\alpha(\mu)(w)(1-|w|^2) \in L^\infty,\ \text{if } \alpha>1.\end{aligned}$$

Notice that $B^\alpha \subset H^\infty$ as $0<\alpha<1$ and using the fact that, as $\alpha>1$, $g\in B^\alpha$ if and only if $(1-|w|^2)^{\alpha-1}g(w)\in L^\infty$, we easily see that (2) is true for all these cases. Thus, T_μ^α is bounded on B^α.

Conversely, if T_μ^α is bounded on B^α, then $T_\mu^\alpha(1) = P_{\alpha-1}(\mu) \in B^\alpha$. If $R_\alpha(\mu) \in L^\infty$, as above, we get that there is a constant $c>0$, independent of g, such that

$$(1-|w|^2)^\alpha|g(w)Q_\alpha(\mu)(w)| \le c\|g\|_{B^\alpha}.$$

For $0<\alpha<1$, we have already shown that $P_{\alpha-1}(\mu)\in B^\alpha$.

For $\alpha \geq 1$, using test functions $g_a(w) = \log \frac{2}{1-\bar{a}w}$ and $g_a(w) = (1-\bar{a}w)^{1-\alpha}$, as $\alpha = 1$ and $\alpha > 1$, respectively, we get

$$Q_\alpha(\mu)(w)(1-|w|^2)\log\frac{2}{1-|w|^2} \in L^\infty \quad \text{if } \alpha = 1; \tag{4}$$

$$Q_\alpha(\mu)(w)(1-|w|^2) \in L^\infty \quad \text{if } \alpha > 1. \tag{5}$$

If $\alpha = 1$, $P_{\alpha-1}(\mu) \in B$ implies that $P_{\alpha-1}(\mu)(w)(1-|w|^2)\log\frac{2}{1-|w|^2} \in L^\infty$. Thus, using (3) and (4), we easily see that $P_{\alpha-1}(\mu)'(w)(1-|w|^2)\log\frac{2}{1-|w|^2} \in L^\infty$.

If $\alpha > 1$, then $P_{\alpha-1}(\mu) \in B^\alpha$ implies that

$$P_{\alpha-1}(\mu)(w)(1-|w|^2)^{\alpha-1} \in L^\infty. \tag{6}$$

For $\alpha - 1 \leq 1$, this implies that $P_{\alpha-1}(\mu)(w)(1-|w|^2) \in L^\infty$, and using (3) and (5), we get that $P_{\alpha-1}(\mu) \in B$.

For $\alpha - 1 > 1$, combining (3), (5) and (6) we get $P_{\alpha-1}(\mu)'(w)(1-|w|^2)^{\alpha-1} \in L^\infty$, which is equivalent to

$$P_{\alpha-1}(\mu)(w)(1-|w|^2)^{\alpha-2} \in L^\infty. \tag{7}$$

If $\alpha - 2 > 1$, combining (3), (5) and (7) we get that $P_{\alpha-1}(\mu)'(w)(1-|w|^2)^{\alpha-2}$ is in L^∞, and we continue the process from above until we reach a positive integer n such that $\alpha - n \leq 1$ and $P_{\alpha-1}(\mu)'(w)(1-|w|^2)^{\alpha-n} \in L^\infty$.

Then, $P_{\alpha-1}(\mu)'(w)(1-|w|^2) \in L^\infty$, and so $P_{\alpha-1}(\mu) \in B$.

This completes the proof. □

Note that in the case $d\mu(z) = f(z)dA(z)$, with $f \in B^\alpha$, it follows from part (ii) of Proposition 2.1 that $R_\alpha(\mu)(w) = wf(w)$. Thus μ satisfies the condition R_α if and only if $f \in H^\infty$.

Corollary 2.1. *Let $f \in L^\infty$ and let $T_f^\alpha(g) = P_{\alpha-1}(fg)$ for $g \in B^\alpha$. Then we have:*

(i) *If $0 < \alpha < 1$, then T_f^α is bounded on B^α if and only if $P_{\alpha-1}(f) \in B^\alpha$.*
(ii) *If $\alpha = 1$, then T_f^α is bounded on B^α if and only if $P_{\alpha-1}(f) \in LB$.*
(iii) *If $\alpha > 1$, then T_f^α is bounded on B^α if and only if $P_{\alpha-1}(f) \in B$.*

Proof. Follows from Proposition 2.1 and Theorem 2.1. □

Since the Bloch space B is the dual of L_a^1 and $T_\mu^* = T_{\bar{\mu}}$ under this duality, we get the following corollary. Similar results including a few other restrictions can be found in [4] and [8].

Corollary 2.2. *Let μ be a complex measure satisfying condition R_α. Then $T_{\bar{\mu}}$ is bounded on L_a^1 if and only if $P(\mu) \in LB$.*

Remark. The proof of Theorem 2.1 implies that the condition R_α in the theorem can be replaced by a more general condition involving Carleson measures.

Recall that for a positive Borel measure ν on $\mathbb{D}$ we say that ν is a Carleson measure on the Bergman space if there exists $c > 0$ such that, for all $h \in L_a^2$,

$$\int_{\mathbb{D}} |h(z)|^2 d\nu(z) \leq c \int_{\mathbb{D}} |h(z)|^2 dA(z).$$

Let $\nu_\alpha(\mu)$ be the positive measure defined by $d\nu_\alpha(\mu)(z) = |R_\alpha(\mu)(z)|dA(z)$.

If μ satisfies condition R_α, then it is easy to see that $\nu_\alpha(\mu)$ is a Carleson measure. By inspecting the proof of Theorem 2.1 we can see that, whenever $R_\alpha(\mu)$ is in L^1, the statements (i), (ii) and (iii) of Theorem 2.1 are still true if the condition R_α is replaced by a weaker one, namely, by requiring only that the measure $\nu_\alpha(\mu)$ is a Carleson measure for the Bergman space.

Note that the measure $\nu_\alpha(\mu)$ is a Carleson measure for the Bergman space if and only if there exists $r > 0$, such that

$$\sup_{z\in\mathbb{D}} \frac{1}{|D(z,r)|} \int_{D(z,r)} |R_\alpha(\mu)(w)| dA(w) < \infty,$$

where $D(z,r)$ denotes the hyperbolic disk with center z and radius r. See [9, Theorem 6.2.2] for details.

Next we give an example of an unbounded L^1 function that induces a bounded Toeplitz operator on B^α. We present part of the calculations as a separate lemma that covers a wider class of functions.

Lemma 2.2. *Let $f(z) = \sum_{n=0}^\infty a_n(1-|z|^2)^n$ be a function in $L^1((1-|z|^2)^{\alpha-1}dA(z))$, for $\alpha > 0$, and such that $\sum_{n=0}^\infty \frac{|a_n|}{n+\alpha}$ is finite. Let $d_\alpha = \alpha\sum_{n=0}^\infty \frac{a_n}{n+\alpha}$ and let $F_\alpha(f) = \int_{\mathbb{D}} f(z)(1-|z|^2)^{\alpha-1}dA(z)$. Then, for $d\mu(z) = f(z)dA(z)$ we have*

$$R_\alpha(\mu)(w) = d_\alpha w - \frac{\alpha}{\bar w}\left(F_\alpha(f) - \frac{1}{(1-|w|^2)^\alpha}\int_{D_w} f(z)(1-|z|^2)^{\alpha-1}dA(z)\right),$$

where $D_w = \{z : |w| < |z| < 1\}$.

Proof. As in the proof of part (ii) of Proposition 2.1 we have that, for $w \neq 0$,

$$R_\alpha(\mu)(w) = wP_{\alpha-1}(f)(w) - \alpha\int_{\mathbb{D}} \frac{(1-|u|^2)^{\alpha-1}}{\bar u(1-\bar w u)^{\alpha+1}} f\circ\psi_w(u)dA(u).$$

Using the given series expansion of f and the identity $1-|\psi_w(u)|^2 = \frac{(1-|w|^2)(1-|u|^2)}{|1-\bar w u|^2}$, we get that $P_{\alpha-1}(f)(w) = \alpha\sum_{n=0}^\infty \frac{a_n}{n+\alpha} = d_\alpha$, and that the above integral equals

$$\int_{\mathbb{D}} \sum_{n=0}^\infty a_n(1-|w|^2)^n \frac{(1-|u|^2)^{n+\alpha-1}}{(1-\bar w u)^{n+\alpha+1}} \frac{(1-\bar u w)^{-n}}{\bar u} dA(u).$$

Since $\frac{(1-\bar u w)^{-n}-1}{\bar u}$ is a conjugate analytic, integrable function, we have that

$$(n+\alpha)\int_{\mathbb{D}} \frac{(1-\bar u w)^{-n}-1}{\bar u} \frac{(1-|u|^2)^{n+\alpha-1}}{(1-\bar w u)^{n+\alpha+1}} dA(u) = \frac{(1-|w|^2)^{-n}-1}{\bar u}.$$

Since also $\int_{\mathbb{D}} \frac{(1-|u|^2)^{n+\alpha-1}}{\bar u(1-\bar w u)^{n+\alpha+1}} dA(u) = 0$, we have that, furthermore,

$$\begin{aligned} R_\alpha(\mu)(w) &= d_\alpha w - \alpha\sum_{n=1}^\infty \frac{a_n}{n+\alpha}(1-|w|^2)^n \frac{(1-|w|^2)^{-n}-1}{\bar w} \\ &= d_\alpha w - \frac{\alpha}{\bar w}\left(\sum_{n=0}^\infty \frac{a_n}{n+\alpha} - \sum_{n=0}^\infty \frac{a_n}{n+\alpha}(1-|w|^2)^n\right) \\ &= d_\alpha w - \frac{\alpha}{\bar w}\left(F_\alpha(f) - \frac{1}{(1-|w|^2)^\alpha}\int_{D_w} f(z)(1-|z|^2)^{\alpha-1}dA(z)\right). \end{aligned}$$

□

Example. For $\alpha > 0$, let $c_\alpha = \frac{1}{\alpha}\left(\sum_{n=1}^{\infty}\frac{1}{n(n+\alpha)}\right)^{-1}$ and for $z \neq 0$, let

$$f_\alpha(z) = 1 - c_\alpha \sum_{n=1}^{\infty} \frac{1}{n}(1-|z|^2)^n = 1 + c_\alpha \log(|z|^2).$$

The function f_α is a radial, unbounded, L^1 function. Since $P_{\alpha-1}(f_\alpha)$ is a radial analytic function, it must be a constant. It is easy to check that $P_{\alpha-1}(f_\alpha)(0) = 0$ and so $P_{\alpha-1}(f_\alpha) = 0$.

In order to show that T_{f_α} is bounded on B^α we only have to prove that μ_α, with $d\mu_\alpha(z) = f_\alpha(z)dA(z)$, satisfies the condition R_α. Using Lemma 2.2 and the fact that $d_\alpha = 0$, we get that

$$R_\alpha(\mu_\alpha)(w) = \frac{1}{\bar{w}}\left(1 + \alpha c_\alpha \frac{1}{(1-|w|^2)^\alpha}\int_{|w|^2}^{1}(1-t)^{\alpha-1}\log t dt\right).$$

It is easy to see that $\lim_{|w|\to 1}|R_\alpha(\mu)(w)| = 1$. Since $\int_0^1 (1-t)^{\alpha-1}\log t dt = \frac{-1}{\alpha c_\alpha}$, we also have that $\lim_{|w|\to 0}|R_\alpha(\mu)(w)| = \alpha$. Thus, $R_\alpha(\mu_\alpha) \in L^\infty$, and by Theorem 2.1, T_{f_α} is bounded on B^α.

Note that we can also use Lemma 2.2 to show that $R_\alpha(\mu) \in L^\infty$ is not a necessary condition for the boundedness of the Toeplitz operator T_μ. For example, let $d\mu(z) = f(z)dA(z)$ with $f(z) = \frac{1}{|z|^{3/2}}$. Note that for this radial, positive function $F_\alpha(f)$ is equal to $\frac{\Gamma(1/4)\Gamma(\alpha)}{\Gamma(\alpha+1/4)}$, and d_α is equivalent to $\alpha\sum_{n=1}^{\infty}\frac{1}{n^{1/4}(n+\alpha)}$. Using Lemma 2.2 we can get that $|R_\alpha(\mu)(w)| \geq \frac{c}{|w|^{\frac{1}{2}}}$ and so $R_\alpha(\mu)$ is not in L^∞. Since the measure $\nu_\alpha(\mu)$ is a Carleson measure, T_μ is still bounded on B^α.

3. Compact Toeplitz operators

For the next result, we need the following lemma from [5].

Lemma 3.1. *Let $0 < \alpha < 1$ and let T be a bounded linear operator from B^α into a normed linear space Y. Then T is compact if and only if $\|Tg_n\|_Y \to 0$, whenever (g_n) is a bounded sequence in B^α that converges to 0 uniformly on $\overline{\mathbb{D}}$.*

Theorem 3.1. *Suppose*

$$\lim_{|w|\to 1} R_\alpha(\mu)(w) = 0. \tag{8}$$

Then we have:

(i) *If $0 < \alpha < 1$, then T_μ^α is compact on B^α if and only if $P_{\alpha-1}(\mu) \in B^\alpha$.*
(ii) *If $\alpha = 1$, then T_μ^α is compact on B^α if and only if $P_{\alpha-1}(\mu) \in LB_0$.*
(iii) *If $\alpha > 1$, then T_μ^α is compact on B^α if and only if $P_{\alpha-1}(\mu) \in B_0$.*

Proof. For the case $\alpha = 1$ or $\alpha > 1$, let $\{g_n\}$ be a sequence in B^α such that $\|g_n\|_{B^\alpha} \leq 1$ and $g_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$. Let h be in the unit ball of L^1_a. Similarly, as in the proof of Theorem 2.1, we have

$$\begin{aligned}&\langle h, T^\alpha_\mu(g_n)\rangle_\alpha\\&= \alpha\int_{\mathbb{D}} P_\alpha(h\bar{g}_n)(z)(1-|z|^2)^{\alpha-1}\,d\bar{\mu}(z)\\&\qquad+\alpha\int_{\mathbb{D}}[(I-P_\alpha)(h\bar{g}_n)](z)(1-|z|^2)^{\alpha-1}\,d\bar{\mu}(z)\\&= I_1+I_2.\end{aligned}$$

For $0<r<1$, and $\mathbb{D}_r=\{z\ :\ |z|\le r\}$,

$$\begin{aligned}I_2 &= \alpha\int_{\mathbb{D}}\overline{g_n'(w)}h(w)(1-|w|^2)^\alpha R_\alpha(\mu)(w)\,dA(w)\\&= \alpha\left(\int_{\mathbb{D}_r}+\int_{\mathbb{D}\setminus\mathbb{D}_r}\right)\overline{g_n'(w)}h(w)(1-|w|^2)^\alpha R_\alpha(\mu)(w)\,dA(w)=K_1+K_2.\end{aligned}$$

For a fixed $\varepsilon>0$, using condition (8), let r be sufficiently close to 1 so that $|R_\alpha(\mu)(w)|<\varepsilon$ as $w\in\mathbb{D}\setminus\mathbb{D}_r$. Then $|K_2|\le\varepsilon\|g_n\|_{B^\alpha}\|h\|_1\le\varepsilon$.

Since $g_n(z)\to 0$ as $n\to\infty$, we can choose n big enough so that $|g_n(z)|(1-|w|^2)^\alpha<\varepsilon$. Therefore, $|K_1|\le\varepsilon\|R_\alpha(\mu)\|_\infty\|h\|_1\le\varepsilon\|R_\alpha(\mu)\|_\infty$.

Hence $|I_2|<C\varepsilon$, where C does not depend on h, and so $\lim_{n\to\infty}\sup_{\|h\|_1\le 1}|I_2| = 0$. Thus T^α_μ is compact on B^α if and only if $\sup_{\|h\|_1\le1}|I_1|\to 0$ as $n\to\infty$.

Similarly, as in the proof of Theorem 2.1, we have

$$\begin{aligned}I_1 &= \int_{\mathbb{D}} h(w)\overline{g_n(w)Q_\alpha(\mu)(w)}(1-|w|^2)^\alpha\,dA(w)\\&= \left(\int_{\mathbb{D}_r}+\int_{\mathbb{D}\setminus\mathbb{D}_r}\right)h(w)\overline{g_n(w)Q_\alpha(\mu)(w)}(1-|w|^2)^\alpha\,dA(w)=M_1+M_2.\end{aligned}$$

It is easy to see that if $P_{\alpha-1}(\mu)$ is in LB_0 and B_0 as $\alpha=1$ and $\alpha>1$, respectively, then, as $|w|\to 1$, Q_α satisfies, respectively:

$$\begin{aligned}&Q_\alpha(\mu)(w)(1-|w|^2)\log\frac{2}{1-|w|^2}\to 0,\ \text{if }\alpha=1;\\&Q_\alpha(\mu)(w)(1-|w|^2)\to 0,\ \text{if }\alpha>1.\end{aligned}$$

Again, notice that, as $\alpha>1$, $g\in B^\alpha$ if and only if $(1-|z|^2)^{\alpha-1}g(z)\in L^\infty$, and we may choose r sufficiently close to 1 such that whenever $w\in\mathbb{D}\setminus\mathbb{D}_r$, $|Q_\alpha(\mu)(w)|(1-|w|^2)\log\frac{2}{1-|w|^2}<\varepsilon$ as $\alpha=1$; and $|Q_\alpha(\mu)(w)|(1-|w|^2)^\alpha<\varepsilon$ as $\alpha>1$, respectively.

We see that $|M_2|\le\varepsilon\alpha\|g_n\|_{B^\alpha}\|h\|_1\le C\varepsilon$, where C does not depend on h.

Since $g_n(z)\to 0$ uniformly on compact subsets of $\mathbb{D}$, we can choose n big enough so that $|g_n(w)|<\varepsilon$. Hence we easily see that, as n is big enough, $|M_1|<C\varepsilon$, where C does not depend on h. Therefore, $\sup_{\|h\|_1\le1}|I_1|\to 0$ as $n\to\infty$, and so T^α_μ is compact on B^α as $\alpha=1$ or $\alpha>1$.

Now let $0<\alpha<1$. Let $\{g_n\}$ be a sequence in B^α such that $\|g_n\|_{B^\alpha}\le 1$ and $g_n(z)\to 0$ uniformly on $\mathbb{D}$. Let h be in the unit ball of L^1_a. By a similar discussion as above, T^α_μ is compact on B^α if and only if $\sup_{\|h\|_1\le1}|I_1|\to 0$ as $n\to\infty$. However, since $\|g_n\|_{B^\alpha} \leq 1$ and $g_n(z) \to 0$ uniformly on $\overline{\mathbb{D}}$, we can choose n big enough such that $|g_n(w)| < \varepsilon$ uniformly for $w \in \overline{\mathbb{D}}$.

Thus, as n is big enough, $|I_1| \leq \varepsilon\alpha\|P_{\alpha-1}(\mu)\|_{B^\alpha}\|h\|_1$.

Therefore, $\sup_{\|h\|_1\leq 1}|I_1| \to 0$ as $n \to \infty$, and so T_μ^α is compact on B^α.

Conversely, if T_μ^α is compact on B^α. Suppose first that $\alpha = 1$ or $\alpha > 1$. Then $\sup_{\|h\|_1\leq 1}|I_1| \to 0$ as $n \to \infty$ for any sequence g_n in B^α such that $\|g_n\|_{B^\alpha} \leq 1$ and $g_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$.

Let $h_z(w) = \dfrac{(1-|z|^2)^\alpha}{(1-\bar{z}w)^{2+\alpha}}$. Then $\|h_z\|_1 \leq C$, and

$$\begin{aligned} I_1 &= (1-|z|^2)^\alpha \int_{\mathbb{D}} \frac{\overline{g_n(w)Q_\alpha(\mu)(w)}(1-|w|^2)^\alpha}{(1-\bar{z}w)^{2+\alpha}}\, dA(w) \\ &= (\alpha+1)^{-1}(1-|z|^2)^\alpha\overline{g_n(z)Q_\alpha(\mu)(z)}. \end{aligned}$$

Thus, $\sup_{\|h\|_1\leq 1}|I_1| \to 0$ as $n \to \infty$ implies

$$\lim_{n\to\infty}\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g_n(z)||Q_\alpha(\mu)(z)| = 0. \tag{9}$$

As $\alpha = 1$ or $\alpha > 1$, using testing functions $g_a(z) = \left(\log\frac{1}{1-|a|^2}\right)^{-1}\left(\log\frac{2}{1-\bar{a}z}\right)^2$ and $g_a(z) = (1-|a|^2)(1-\bar{a}z)^{-\alpha}$ respectively in (9), as in the proof of Theorem 2.1, we easily see that $P_{\alpha-1}(\mu)$ is in LB_0 and B_0, respectively.

Now let $0 < \alpha < 1$ and let T_μ^α be compact on B^α. Then T_μ^α is bounded on B^α. By Theorem 2.1, $P_{\alpha-1}(\mu) \in B^\alpha$. The proof is completed. □

Remark. The requirement that $\lim_{|w|\to 1} R_\alpha(\mu)(w) = 0$, in the proof of Theorem 3.1, can be replaced by a more general requirement that the measure $\nu_\alpha(\mu)$, defined in Section 2, is a compact Carleson measure.

Recall that for a positive Borel measure ν on $\mathbb{D}$ we say that ν is a compact Carleson measure whenever

$$\lim_{|z|\to 1}\frac{1}{|D(z,r)|}\int_{D(z,r)} d\nu(w) = 0,$$

where $D(z,r)$ denotes the hyperbolic disk with center z and radius r. See [9, Theorem 6.2.5] for more details. If the measure μ is such that $\lim_{|w|\to 1} R_\alpha(\mu)(w) = 0$, then it is easy to see that $\nu_\alpha(\mu)$ is a compact Carleson measure.

Note that in the case $d\mu(z) = f(z)dA(z)$, with $f \in B^\alpha$, it follows from part (ii) of Proposition 2.1 that $R_\alpha(\mu)(w) = wf(w)$. Thus the condition (8) in Theorem 3.1 is satisfied only when $f = 0$.

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