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BMOA AND Ba SPACES ON COMPACT BORDERED RIEMANN SURFACES*

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In [1], Metzger proposed whether John-Nirenberg's theorem for BMOA in the unit disk can be translated to Riemann surfaces. For the compact bordered Riemann surface R we give an affirmative answer. Also we introduce a special class of Ba spaces on R and then point out a relationship between BMOA(R) and Ba(R).

Let Δ be the unit disk: $\Delta = \{z : |z| < 1\}$. Then [2]

$$\text{BMOA}(\Delta) = \left\{ f, f \text{ is analytic in } \Delta, \|f\|_* = \sup_{a \in \Delta} \left(\int_{\partial\Delta} |f(e^{i\theta}) - f(a)| d\mu_a(\theta) \right) < \infty \right\},$$

$$\text{where } d\mu_a(\theta) = \frac{1 - |a|^2}{|e^{i\theta} - a|^2} \frac{d\theta}{2\pi}.$$

For BMOA(Δ), we know the following important John-Nirenberg's theorem [2,3].

Theorem A. Let $f(z)$ be an analytic function in Δ , $f(z)$ belong to BMOA(Δ) iff

$$\mu_a(\{e^{i\theta} \in \partial\Delta, |f(e^{i\theta}) - f(a)| > t\}) \leq Ke^{-\beta t}, \quad (2.1)$$

where K and β are two constants, and when $f \in \text{BMOA}(\Delta)$, $\beta = c/\|f\|_*$, c is a constant, $\mu_a(E)$ is the harmonic measure of E with respect to Δ .

In [1], Metzger defined the BMOA(R) in the following way.

Definition 1. Let R be a Riemann surface which possesses a harmonic Green's function, denoted by $g_R(q, q_0)$. Then a holomorphic function on R is said to belong to BMOA(R) if

$$(\mathbf{B}_R(F))^2 = \sup_{q_0 \in R} \left\{ \int_R \int_R |F'(q)|^2 g_R(q, q_0) dq dq^- \right\} < \infty. \quad (2.2)$$

Theorem 1. Let R be a compact bordered Riemann surface whose boundary ∂R consists of finite numbers of analytic Jordan curves, $F(q)$ be a holomorphic function on R . Put

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$$\tilde{E}_0(t) = \{ q \in \partial R, |F(q) - F(q_0)| > t, q_0 \in R \}.$$

Then, $F(q) \in \text{BMOA}(R)$ if and only if

$$\tilde{\mu}(\tilde{E}_0(t)) = \sup_{q_0 \in R} \left\{ \int_{\tilde{E}_0(t)} \frac{\partial g_R(q, q_0)}{\partial n} ds \right\} \leq K e^{-\beta t}, \quad (2.3)$$

where n is the inner normal at q , K and β are two constants, and when $F(q) \in \text{BMOA}(R)$, then $\beta = c/B_R(F)$, c is a constant.

To prove Theorem 1, we need the following facts: For a compact bordered Riemann surface R , we know that the universal covering surface for R is conformally equivalent to the unit disk Δ , the group of deck transformations is a Fuchsian group $\Gamma = \{\gamma_n\}$. We denote by Ω and Δ/Γ its fundamental region and the associated Riemann surface respectively. Now if F is a holomorphic function on R , then $f(z) = F \circ \varphi(z)$ is its "pull back", where $\varphi: \Delta \rightarrow R$ is a universal covering map (or projective map). It follows immediately that $f(z)$ is an automorphic function with respect to Γ . And we define the "pull back" of $\text{BMOA}(R)$ as follows:

$$\text{BMOA}(\Delta/\Gamma) = \text{BMOA}(\Delta) \cap \{ \text{automorphic functions with respect to } \Gamma \}.$$

Metzger proved

Theorem B. $f(z) \in \text{BMOA}(\Delta/\Gamma)$ if and only if $F \in \text{BMOA}(R)$.

The following theorem will be used.

Theorem C^[4]. Let R be a compact bordered Riemann surface whose Green function $g_R(q, q_0)$ can be lifted to Δ via the projection $\varphi(z)$, i.e. $g_\Gamma(z, a) = g_R(\varphi(z), \varphi(a))$. Then,

$$(i) \quad g_\Gamma(z, a) = \sum_{\gamma_n \in \Gamma} g_\Delta(z, \gamma_n(a)) = \sum_{n=0}^{\infty} g_\Delta(z, a_n) \text{ where } a_n = \gamma_n(a), a_0 = a \text{ and}$$

$$g_\Delta(z, a) = \log \left| \frac{1 - \bar{a}z}{z - a} \right|;$$

$$(ii) \quad \frac{\partial g_\Gamma(z, a)}{\partial \gamma} = \sum_{n=0}^{\infty} \frac{1 - |a_n|^2}{|z - a_n|^2}, \quad z = e^{i\theta};$$

$$(iii) \quad \int_{\partial\Omega \cap \partial\Delta} \frac{\partial g_\Gamma(z, a)}{\partial \gamma} ds = 2\pi, \quad z = e^{i\theta},$$

where γ is the inner normal at $z \in \partial\Omega \cap \partial\Delta$.

Proof of Theorem 1. We suppose that F belongs to $\text{BMOA}(R)$. Let $\varphi: \Delta \rightarrow R$ be a projective mapping with $\varphi(a) = q_0$, $a \in \Omega$. Since $\varphi: \Delta \rightarrow R$ is one to one, $\varphi: \bar{\Omega} \rightarrow R$ is surjective and $\varphi(\partial\Omega \cap \partial\Delta) = \partial R$. By the definition, it holds that

$$\int_{E_0(t)} \frac{\partial g_R(q, q_0)}{\partial n} ds = \int_{E_0(t)} \frac{\partial g_\Gamma(z, a)}{\partial \gamma} ds, \tag{2.4}$$

where $E_0(t) = \{e^{i\theta} \in \partial\Omega \cap \partial\Delta, |f(e^{i\theta}) - f(a)| > t\}$. Putting $\gamma_n(E_0(t)) = E_n(t)$, $\gamma_n \in \Gamma$, we have

$$\bigcup_{n=0}^{\infty} E_n(t) = \{e^{i\theta} \in \partial\Delta, |f(e^{i\theta}) - f(a)| > t\} = E(t).$$

Since $g_\Delta(z, \gamma_n, (a)) = g_\Delta(\gamma_n^{-1}(z), a)$, we have $dg_\Delta^*(z, \gamma_n(a)) = dg_\Delta^*(\gamma_n^{-1}(z), a)$, where $g_\Delta^*(z, a)$ is the conjugate function of $g_\Delta(z, a)$. Therefore,

$$\begin{aligned} \int_{E_0(t)} \frac{\partial g_\Delta(z, \gamma_n, (a))}{\partial \gamma} ds &= \int_{E_0(t)} dg_\Delta^*(z, \gamma_n(a)) = \int_{E_0(t)} dg_\Delta^*(\gamma_n^{-1}(z), a) \\ &= \int_{\gamma_n^{-1}(E_0(t))} \frac{1 - |a|^2}{(z - a)^2} d\theta, \quad z = e^{i\theta}. \end{aligned}$$

According to Theorems A and C(ii), we get

$$\begin{aligned} \frac{1}{2\pi} \int_{E_0(t)} \frac{\partial g_\Gamma(z, a)}{\partial \gamma} ds &= \sum_{n=0}^{\infty} \int_{E_0(t)} \frac{1 - |a_n|^2}{|z - a_n|^2} \frac{d\theta}{2\pi} = \int_{\bigcup_{n=0}^{\infty} E_n(t)} \frac{1 - |a|^2}{|z - a|^2} \frac{d\theta}{2\pi} \\ &= \int_{E(t)} d\mu_a(\theta) \leq Ke^{-\beta t}, \end{aligned} \tag{2.5}$$

where $\beta = c / \|f\|_*$. But we know that^[1]

$$\|f\|_* \sim B_\Delta(f) = \sup_{a \in \Delta} \left\{ \iint_\Delta |f'(z)|^2 g_\Delta(z, a) dx dy \right\},$$

therefore $\beta = c' / B_\Delta(f)$.

On the other hand, by Theorem C(i)

$$\begin{aligned} \iint_R |F'(q)|^2 g_R(q, q_0) dq d\bar{q} &= \iint_\Omega |f'(z)|^2 g_\Gamma(z, a) dx dy \\ &= \iint_\Omega |f'(z)|^2 \sum_{n=0}^{\infty} g_\Delta(z, a_n) dx dy = \int_\Delta \int_\Omega |f'(z)|^2 g_\Delta(z, a) dx dy. \end{aligned} \tag{2.6}$$

Note that the supremum over a in Δ or the supremum over a in Ω is the same, i. e. $B_\Delta(f) = B_R(F)$. Combining (2.6) with (2.5) and (2.4), we obtain (2.3). Conversely, according to the above discussion, if (2.3) holds, then

$$\mu_a(\{e^{i\theta} \in \partial\Delta, |f(e^{i\theta}) - f(a)| > t\}) = \frac{1}{2\pi} \int_{E_0(t)} \frac{\partial g_R(q, q_0)}{\partial n} ds \leq Ke^{-\beta t}.$$

By Theorem A, $f(z) \in \text{BMOA}(\Delta)$. On the other hand, it is obvious that $f(z)$ is an automorphic function with respect to Γ , and by Theorem B, $f(z) \in \text{BMOA}(\Delta/\Gamma)$, i. e. $F \in \text{BMOA}(R)$.

Similar to the proof in the case of the unit disk and noting Theorem C(iii), we have

Corollary 1. $F \in \text{BMOA}(R)$ if and only if

$$\sup_{q_0 \in R} \left(\int_{\partial R} |F(q) - F(q_0)|^p \frac{\partial g_R(q, q_0)}{\partial n} ds \right) = M_p < \infty \quad (2.7)$$

and $M_p^{1/p} \sim B_R(F)$. When $F \in \text{BMOA}(R)$, $M_p \leq \frac{K\Gamma(p+1)}{c^p} (B_R(F))^p$.

In [5], a new function space called a Ba space is introduced. Here we shall give a special class of Ba space on a compact bordered Riemann surface, denoted by $H_{\text{Ba}}(R)$.

First, let us recall the definition of Hardy class on the Riemann surface

$$H_p(R) = \{ F, F \text{ is holomorphic functions on } R \text{ and } |F(q)|^p \text{ has a harmonic majorant} \}.$$

If $H(q)$ is the least harmonic majorant of $|F(q)|^p$, then $\|F\|_{H_p} = |H(q_0)|^{1/p}$, $q_0 \in R$, and $H_p(R)$ is a Banach space with the norm $\|\cdot\|_{H_p}$, $p \geq 1$ [6]. Given $H_p(R)$, let $H_p(\Delta/\Gamma) = H_p(\Delta) \cap \{ \text{automorphic functions with respect to } \Gamma \}$ be its "pull back". It is easy to see that if $H(q)$ is the least harmonic majorant of $|F(q)|^p$, then $h(z) = H \cdot \varphi(z)$ is the least harmonic majorant of $|f(z)|^p$, and $h(z)$ is automorphic with respect to Γ and $\|f\|_{H_p} = |h(a)|^{1/p}$.

Now we give the definition of Ba space on the Riemann surface as follows. Let $E(z) = \sum_{m=1}^{\infty} a_m z^m$ be an entire function with finite order $0 < \rho < \infty$ and mean type $\sigma < \infty$, and $a_m \geq 0$. Let $\{p_m\}$ be a sequence satisfying

$$1 \leq p_1 < p_2 < \dots < p_m \rightarrow \infty$$

and

$$\overline{\lim}_{m \rightarrow \infty} p_m / m^{1/\rho} = p^* < \infty. \quad (3.1)$$

For $F(q) \in \bigcap_{m=1}^{\infty} H_{p_m}(R)$, set

$$I(F, \alpha) = \sum_{m=1}^{\infty} a_m \|F\|_{H_{p_m}}^m \alpha^m. \quad (3.2)$$

We denote by d_F the radius of convergence of (3.2). Then

$$H_{\text{Ba}}(R) = \{ F, F \in \bigcap_{m=1}^{\infty} H_{p_m}(R), d_F > 0 \},$$

which is a Banach space with the norm $\|\cdot\|_{\text{Ba}}$, defined by

$$\|F\|_{Ba} = \inf \left\{ \frac{1}{|\alpha|}, I(F, |\alpha|) \leq 1 \right\}.$$

Putting $\|F\| = \sup_{q_0 \in R} \{ \|F(q) - F(q_0)\|_{Ba} \}$, we have

Theorem 2. *There exists a constant c such that*

$$c^{-1}B_R(F) \leq \|F\| \leq cB_R(F). \tag{3.3}$$

Proof. We denote $F(q) - F(q_0)$ by $\hat{F}(q)$. Then by Corollary 1 we have

$$\|\hat{F}\|_{Hp_m}^{p_m} = \int_{\partial R} |F(q) - F(q_0)|^{p_m} \frac{\partial g_R(q, q_0)}{\partial n} ds \leq \frac{K\Gamma(p_m + 1)}{C^{p_m}} (B_R(F))^{p_m},$$

and therefore

$$I(\hat{F}, \frac{|\alpha|}{B_R(F)}) = \sum_{m=1}^{\infty} a_m \|\hat{F}\|_{Hp_m}^m \left(\frac{|\alpha|}{B_R(F)} \right)^m \leq \sum_{m=1}^{\infty} A_m |\alpha|^m, \tag{3.4}$$

where $A = aK \frac{m}{p_m} (\Gamma(p + 1))^{m/p_m} / c^m$. By Stirling's formula

$$\Gamma(p + 1) = p^p e^{-p} \sqrt{2\pi p} (1 + o(1))$$

and the following relation

$$(\rho e \sigma)^{\frac{1}{\rho}} = \overline{\lim}_{m \rightarrow \infty} (m^{\frac{1}{\rho}} \sqrt[m]{a_m}),$$

noting the condition (3.1), we can show that $\overline{\lim}_{m \rightarrow \infty} \sqrt[m]{A_m} \leq p^*(\rho e \sigma)^{1/\rho} / ce$, then the radius of convergence of the series (3.4) $d_F > \alpha_0 = ec / 2p^*(\rho e \sigma)^{1/\rho} > 0$, and therefore

$$I(\hat{F}, \frac{|\alpha_0|}{B_R(F)}) \leq K(\rho, \sigma, p^*) < \infty.$$

Similar to the proof in [4], we get

$$\|F\| \leq (\max \{ \frac{1}{\alpha_0}, \frac{1}{\alpha_0} K(\rho, \sigma, p^*) \}) B_R(F).$$

Conversely we take $c_0 = \min_m \{ \frac{1}{\sqrt[m]{a_m}} \} = \frac{1}{\sqrt[m_0]{a_{m_0}}}$, and then

$$1 \geq I(\hat{F}, \frac{1}{\|\hat{F}\|_{Ba}}) \geq a_{m_0} \|\hat{F}\|_{Hp_{m_0}}^{m_0} / \|\hat{F}\|_{Ba}^{m_0}.$$

Thus,

$$\|\hat{F}\|_{Hp_{m_0}} \leq c_0 \|F\|.$$

By Corollary 1 we get

$$B_R(F) \leq c \|F\|.$$

discussion .

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